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Session 5 Fractal Weyl laws

Summary of previous session:

$$f(z) = z^2 + c \qquad c < -2$$

$$g_1(z) = \sqrt{z-c} \qquad \text{inverse brances}$$
$$g_2(z) = -\sqrt{z-c}$$

$$D_1, D_2 \subseteq \mathbb{C} \qquad \overline{g_i(D_j)} \subseteq D_i$$

$$J = \bigcup_{i=1}^{2} g_i(J) \qquad J: \text{attractor of the IFS}$$

Coding

$$\begin{tikzcd} \Sigma \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \Sigma \arrow[d, "\pi"] \\ J \arrow[r] & J \end{tikzcd} \qquad \Sigma = \{0,1\}^{\mathbb{N}}$$

$$\pi(\underline{i}) = \lim_{n\to\infty} g_{i_1} \circ \dots \circ g_{i_n}(0)$$

Transfer operator

$$L(s)u(z) = \sum_{i=1}^{2} [g_i'(z)]^{-s} u(g_i(z)) \qquad z \in \mathbb{C}$$

$D = D_1 \cup D_2$

acting on $H^2(D) = \{ u \text{ holom in } D, \iint_D |u(z)|^2 d\mu(z) < \infty \}$

Spectral properties of $L(s)$

Prop 1:
$L(s)$ is trace class and

$$\det(I - L(s)) = \prod_{i=1}^{\infty} (1 - \lambda_i(L(s))) \leq \exp(c|s|^2)$$

Prop 2:

$$\det(I - L(s)) = \exp\left(- \sum_{n=1}^{\infty} \frac{1}{n} \sum_{f^n z = z} \frac{[(f^n)'(z)]^{-s}}{1 - [(f^n)'(z)]^{-1}} \right)$$

Prop 3: Let $J \subseteq \mathbb{R}$ be the Julia set of f_c.
There exist constants δ_0 and $K = K(c)$ st for $\delta < \delta_0$, the connected component of
$J + [-\delta, \delta]$

have length at most $K\delta$.

For this, we will prove that J is quasi-self similar:

~~Session~~ S1

Lemma: Let $J \subseteq \mathbb{R}$ be the Julia set for f_c. J is quasi-self similar, ie, there exists $c>0$ and $r_0>0$ st for $x_0 \in J$ and $r<r_0$, there exists a map

$$g:[x_0-r,\ x_0+r] \longrightarrow \mathbb{R} \qquad \text{s.t.}$$

$$g(J \cap [x_0-r,\ x_0+r]) \subset J$$

$$c r^{-1}|x-y| \leq |g(x)-g(y)| \leq c^{-1} r^{-1} |x-y| \qquad x,y \in [x_0-r,\ x_0+r]$$

Proof

We had our intervals D_1, D_2 and maps g_1 and g_2. Define, for $\underline{i}=(i_1, \dots i_k) \in \{1,2\}^k$

$$g_{i_1 \dots i_k} = g_{i_k} \circ \dots \circ g_{i_1}$$

So $g_{i_1 \dots i_k} : D \to D$.

By continuity and compactness, there exist C_{min}, C_{max} st

$$C_{min} \leq |g_1'|, |g_2'| \leq C_{max}$$

If we define

$$D_{i_1 \dots i_k} = g_{i_k} \circ \dots \circ g_{i_1}(D)$$

Then we have

$$C_{min} |\cancel{D}_{i_1 \dots i_k}| \leq |D_{i_1 \dots i_k s}| \leq C_{max} |D_{i_1 \dots i_k}|$$

Recall that our map has bounded distortion:

$$f^k : D_{i_1 \dots i_k} \longrightarrow D \qquad (\text{inverse of } g_{i_1 \dots i_k})$$

is such that $\exists$ b_0, b_1 st

$$b_0^{-1} \leq |D_{i_1 \dots i_k}| \cdot |(f^k)'(x)| \leq b_0 \qquad \forall x \in D_{i_1 \dots i_k}$$

ie, $|D_{i_1 \dots i_k}| \asymp |(f^k)'(x)|^{-1}$.

This can be improved to:

$$b_1^{-1} \cdot |y-z| \leq |f^k(y) - f^k(z)| \cdot |D_{i_1 \dots i_k}| \leq b_1 |y-z|$$

for every $y, z \in D_{i_1 \dots i_k}$.

An important corollary of this

<u>Corollary</u>

Let $d = \text{dist}(D_1, D_2)$, then

Obvious

a) $d b_1^{-1} |D_{i_1 \dots i_k}| \le \operatorname{dist}(D_{i_1 \dots i_k 1}, D_{i_1 \dots i_k 2}) \le |D_{i_1 \dots i_k}|$

b) Let $\lambda = d b_1^{-1} C_{min}$. For all $i = (i_1 \dots i_k)$, if $x \in D_{i_1 \dots i_k} \cap E$ and

$$|D_{i_1 \dots i_k}| \le r < |D_{i_1 \dots i_k}| C_{min}^{-1} \quad \text{then}$$

$$B(x, \lambda r) \cap E \subseteq D_{i_1 \dots i_k} \cap E \subseteq B(x, r)$$

Obvious

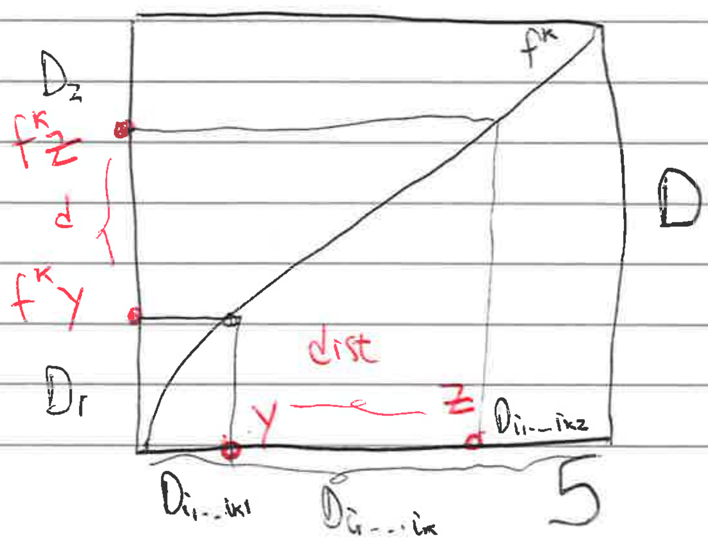
Proof:

a) $f^k : D_{i_1 \dots i_k} \to D$ is a C^1 bijection st

BD $b_1^{-1} |y - z| \le |f^k y - f^k z| \cdot |D_{i_1 \dots i_k}| \le b_1 |y - z|$ $\quad y, z \in D_{i_1 \dots i_k}$

Taking $y \in D_{i_1 \dots i_k 1}$, $z \in D_{i_1 \dots i_k 2}$ st $f^k(y) \in D_1$, $f^k(z) \in D_2$ satisfy

$$d = |f^k y - f^k z|$$



RHS

Using y and z in BD we obtain

$$d\cdot|D_{i_1\dots i_k}| \le b_1|y-z| = b_1\, d(D_{i_1\dots i_{k1}}, D_{i_1\dots i_{k2}})$$

so

$$b_1^{-1} d\cdot|D_{i_1\dots i_k}| \le d(D_{i_1\dots i_{k1}}, D_{i_1\dots i_{k2}})$$

as we wanted. ∎

b) Note that for $(i_1\dots i_k)$ and

$$\lambda r < d b_1^{-1}|D_{i_1\dots i_k}| \qquad \text{(hypothesis)}$$

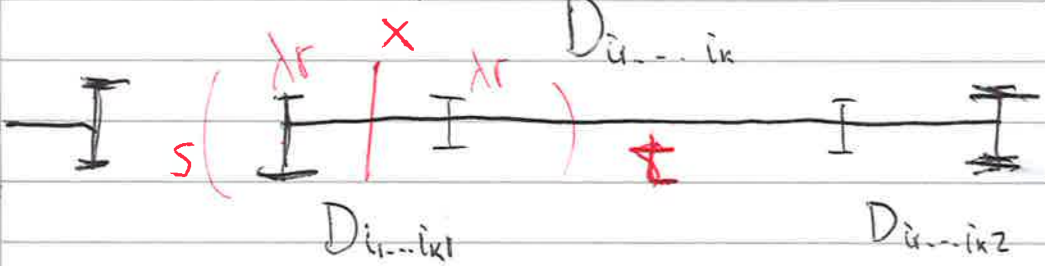
by (a),

$$\lambda r < d(D_{i_1\dots i_{k1}}, D_{i_1\dots i_{k2}}) = t$$

and

$$s = \mathrm{dist}(D_{i_1\dots i_k}, D_{i_1\dots i_k}) \ge d b^{-1}|D_{i_1\dots i_{k-1}}| \ge d b^{-1}|D_{i_1\dots i_k}| > \lambda r$$

In a picture



Then $J \cap B(x, \lambda r) \subseteq D_{i_1 \dots i_k} \cap J$ as we wanted $\square$

Cor (Lemma)
There exist $C > 0$ and $r_0 > 0$ st if $x \in J$, $r < r_0$, there exists $g : B(x,r) \cap J \to J$ st

$$C^{-1} r^{-1} |x-y| \leq |g(x) - g(y)| \leq C r^{-1} |x-y| \quad , \; x, y \in J \cap B(x,r)$$

Proof:
Let $r < r_0 := d b_1^{-1} |D|$ and $x \in J$. Then, by BD

$\exists$ k and $(i_1 \dots i_k)$ st $x \in D_{i_1 \dots i_k}$ and

$$d b_1^{-1} C_{min} |D_{i_1 \dots i_k}| \leq r < d b_1^{-1} |D_{i_1 \dots i_k}$$

$$|D_{i_1 \dots i_k}| \leq \frac{r}{d b_1^{-1} C_{min}} < |D_{i_1 \dots i_k}| C_{min}^{-1}$$

By the previous result,

$$B(x, \frac{r}{d b_1^{-1} C_{min}} \cdot \underbrace{d b_1^{-1} C_{min}}_{\lambda}) \cap J \subseteq D_{i_1 \dots i_k}$$

By BD, $f^k : J \cap B(x,r) \to J$ satisfies

$$b_1^{-1} |y-z| \leq |f^k y - f^k z| \cdot |D_{i_1 \dots i_k}| \leq b_1 |y-z| \quad \text{so}$$

$$C_{min} d b_1^{-2} r^{-1} |y-z| \leq |f^k y - f^k z| \leq d r^{-1} |y-z| \quad \text{as we wanted.}$$

Thm 1

Let $\delta = \dim_H J$ be the Hausdorff dimension of the Julia set associated to f. Then for any C_0, there exists C_1 such that

$$|Z(s)| \leq C_1 \exp(C_1 |s|^{\delta})$$

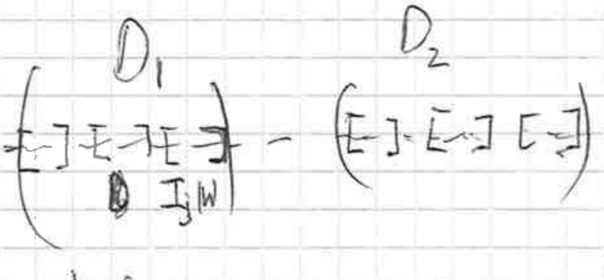
for $|\mathrm{Re}\, s| \leq C_0$.

Sketch of proof:

Put $h = 1/|s|$ for $|\mathrm{Im}\, s|$ large but $|\mathrm{Re}\, s|$ unif. bounded. Decompose J into a union of intervals

$$I_j(h) = (J \cap D_j) + [-h, h] = \bigcup_{p=1}^{P_j(h)} [x_p^j - r_p^j, x_p^j + r_p^j]$$

The intervals $[x_p^j - r_p^j, x_p^j + r_p^j]$ contain the connected components of $I_j(h)$, so by the previous prop, $r_p^j < kh$ as $h \to 0$



Now, define

$$D_{jp}(h) = (x_p^j - r_p^j, x_p^j + r_p^j) + i(-h, h)$$

$$D_j(h) = \bigcup_{p=1}^{P_j} D_{jp}(h)$$

$$D(h) = \bigcup_{j=1}^{2} D_j(h)$$

Classical result (McMullen) $\boxed{\dim_B J = \dim_H J}$

~~$\dim_H J < \infty$~~ $H^{\dim J}(J) < \infty$

and $\dim_H J = \dim_B J$, where $\dim_B J$ is given by

$$\dim_B J = \lim_{r \to 0} \frac{\log N_r(J)}{-\log r}$$

$N_r(J)$: the least number of sets of diameter r that cover J.

So, $N_r(J) \sim r^{-\dim_B(J)}$.

Since the boxes cover J we must have $P_j(h) = O(h^{-\delta})$ Ours is an optimal cover ??

Now we want to establish a bound for $\log|\det(I - \mathcal{L}(s))|$

(   )   (   )

$D_1(h)$ $D_2(h)$

We have that $\mathcal{L}(s)$ decomposes as the sum of two operators, each of these is a sum of two operators

$$\mathcal{L}_1(s)u(z) = \sum_{i=1}^{2} [g_i'(z)]^s u(g_i(z)) \qquad z \in D_1$$

Each of these four operators can be decomposed as a sum of $P_1(h)$ or $P_2(h)$ operators.

$$\mathcal{L}_i = \bigoplus_{t=1}^{P_i(h)} \mathcal{L}^t(s)$$

Using the same procedure as in prop 1, we obtain bounds for the singular values of these operators:

$$\mu_\ell(\mathcal{L}_i^t) \le C\gamma^\ell \quad \text{for some } \gamma \in (0,1)$$

so then

$$\log|\det(I - \mathcal{L}^t(s))| \le \log \prod_{\ell=1}^{\infty} (1 + \mu_\ell(\mathcal{L}^t(s)))$$

$$\le \sum_{\ell=1}^{\infty} (\mu_\ell(\mathcal{L}^t(s)) + O(\mu_\ell^2))$$

$$\le \sum_{\ell\ge 1} C\gamma^\ell$$

$$\log|\det(I-\mathcal{L}(s))| \leq \log \prod_{\ell=1}^{\infty}(1+\mu_\ell(\mathcal{L}(s)))$$

$$\leq \sum_{\ell\geq 1}' (\mu_\ell(\mathcal{L}(s)) + \mathcal{O}(\mu_\ell^2))$$

but

$$\sum_\ell' \mu_\ell\left(\bigoplus_t \mathcal{L}^t\right) = \sum_t' \sum_\ell' \mu_\ell(\mathcal{L}_\ell^t)$$

so

$$\log|\det(I-\mathcal{L}(s))| \leq \sum_t' \sum_\ell' (\mu_\ell(\mathcal{L}^t) + \mathcal{O}(\mu_\ell^2))$$

$$\leq C_1 P_1(h) \sum \gamma^\ell + C_2 P_2(h) \sum \gamma^\ell$$

$$\leq \tilde{C}_1 P_1(h) + \tilde{C}_2 P_2(h)$$

$$= \mathcal{O}(h^{-\delta})$$

as we wanted.

Let $m(s)$ be the multiplicity of the zero s of Z, and

$$n(r,x) = \sum \{ m(s) : |\mathrm{Im}\, s| \le r,\ \mathrm{Re}\, s > x \}$$

so

$$n(r+1,x) - n(r,x) = \sum \{ m(s) : r \le |\mathrm{Im}\, s| \le r+1,\ \mathrm{Re}\, s > x \}$$

Cor

$$n(r+1,x) - n(r,x) \le c_1 r^{\delta} \quad , \ \delta = \dim_H J$$

by summation

$$n(r,x) \le c_2 r^{1+\delta}$$

Proof

We have

$$Z(s) = \exp\left(-\sum_{n \ge 1} \frac{1}{n} \sum_{f^n z = z} \frac{[(f^n)'(z)]^{-s}}{1 - [(f^n)'(z)]^{-1}} \right)$$

We bound $Z(s)$:

- λ expansion factor of f, so

$$|(f^n(z))'| \le \lambda^n \quad \text{so}$$

$$\sum_{f^n z = z} \frac{|(f^n)'(z)|^{-s}}{1 - [(f^n)'(z)]^{-1}} \le \lambda^{-ns} 2^n \quad \text{Then}$$

$$\sum \frac{1}{n} \sum_{f^n z = z} \frac{[(f^n)'(z)]^{-s}}{1 - [(f^n)'(z)]^{-1}} \le \log\left(1 - \frac{2}{\lambda^s}\right)$$

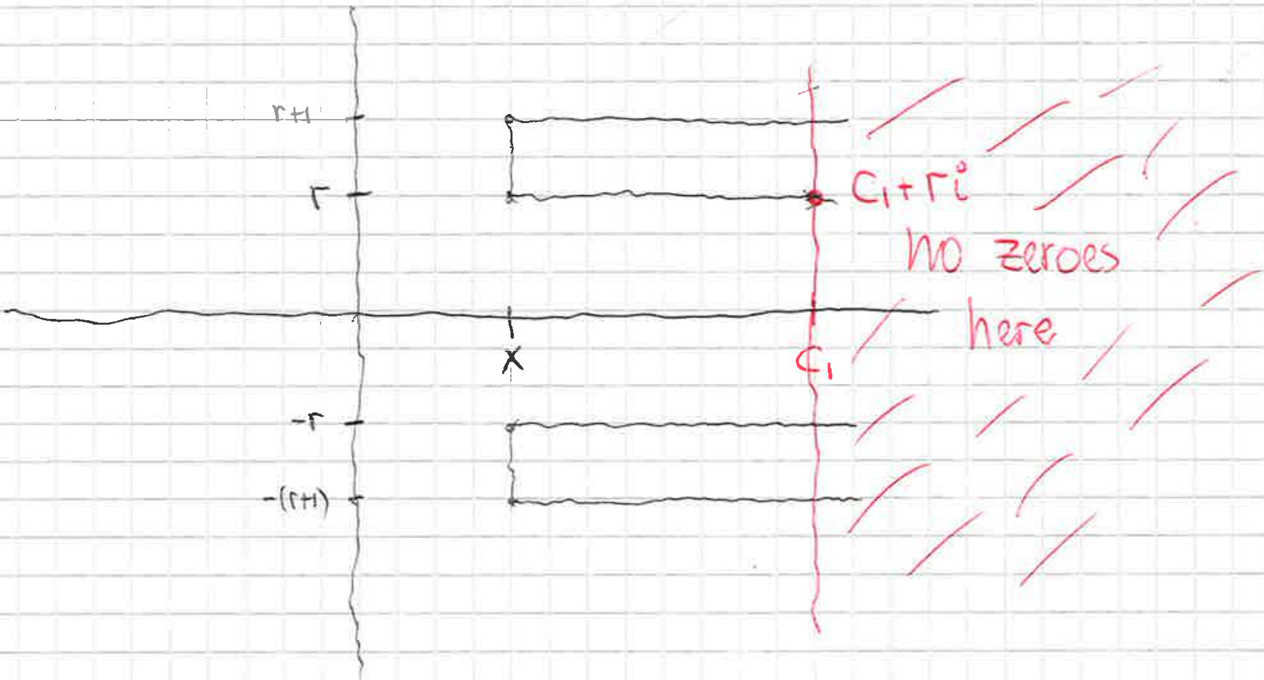
So

$$Z(s) \geq \left(1 - \frac{2}{\lambda^s}\right)$$

So for large s, $|Z(s)| > 1/2$. Now we recall Jensen's formula: for f analytic in $R \supseteq D(0,r)$, if $a_1, \dots, a_n$ are the zeroes of f in $D(0,r)$, then

$$\sum_i \log\left|\frac{r}{|a_i|}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f(Re^{i\theta})}{f(0)}\right| d\theta$$

Now for Z, we want to estimate $n(r+1,x) - n(r,x)$



So As a corollary of Jensen

$$\# \text{zeros} \leq \frac{1}{\log 2} \log \left| \frac{\sup_{|s| \leq 2R} |Z(s)|}{|Z(0)|} \right|$$

$\uparrow$
disco radio R

~~Ex~~ Expansion factor for f : λ

so $(f^n(z))' \leq \lambda^n$

$$\sum_{f^n z = z} \frac{[(f^n)'(z)]^{-s}}{1 - [(f^n)'(z)]} \leq \sum_{f^n z = z} \lambda^{-ns} \leq 2^n \lambda^{-ns}$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{f^n z = z} \frac{[(f^n)'(z)]^{-s}}{1 - [(f^n)'(z)]} \leq \sum_{n \geq 1} \frac{1}{n} 2^n \lambda^{-ns} = \sum_{n \geq 1} \frac{1}{n} \left(\frac{2}{\lambda^s}\right)^n$$

$$\log(1+x) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} (x)^n = -\sum_{n \geq 1} \frac{(-x)^n}{n}$$

so $\displaystyle\sum_{n \geq 1} \frac{1}{n} \left(-\left(-\frac{2}{\lambda^s}\right)\right)^n = -\log\left(1 - \frac{2}{\lambda^s}\right)$

$$\Rightarrow -\sum_{n \geq 1} \frac{1}{n} \sum_{f^n z = z} \frac{[(f^n)'(z)]^{-s}}{1 - [(f^n)'(z)]} \geq \log\left(1 - \frac{2}{\lambda^s}\right)$$

$$Z(s) = \exp\left(-\sum_{n \geq 1} \frac{1}{n} \sum_{f^n z = z} \frac{[(f)^n(z)]^{-s}}{1 - [(f^n)'(z)]}\right) \geq \left(1 - \frac{2}{\lambda^s}\right)$$

For s large, $\cancel{(1 - \frac{2}{\lambda^s})}$ $\left(1 - \frac{2}{\lambda^s}\right) > 1/2$

$$Z(s) = \exp\left(-\sum_{n=1}^{\infty}{}' \frac{1}{n} \sum_{f^n z = z}{}' \frac{[(f^n)'(z)]^{-s}}{1-[(f^n)'(z)]^{-1}}\right) \qquad X = \frac{1}{2+\cancel{X}}$$

and $\qquad \mathrm{Re}\, s \geq c_1 \qquad 2x+x^2=1$

$$\Downarrow$$

$$|Z(s)| > 1/2$$

$$|[g_i'(z)]^s| \leq C \exp(C_1 |s| \cdot |\mathrm{Im}\, z|)$$

$$x^2+2x-1=0$$

Jonsen's formula

f analytic in R st $\mathbb{D} = D(0,r) \subseteq R$, $a_1, \ldots a_n$ zeroes of f in $\mathbb{D}^\circ$ (repeated according multiplicity), $f(0) \neq 0$.

Then

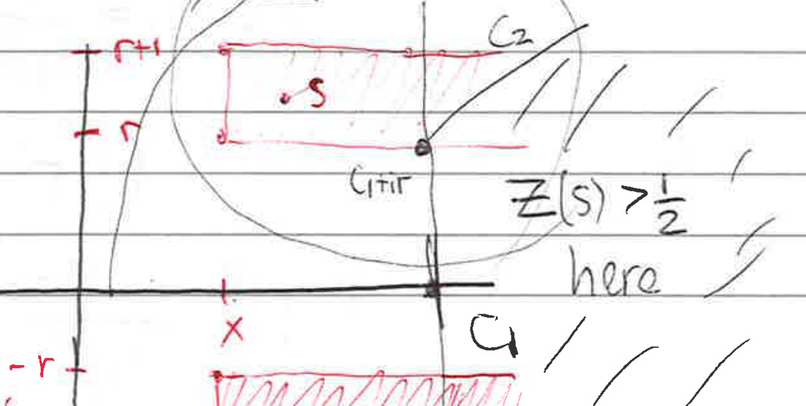
$$\log|f(0)| = \sum_{k=1}^{n} \log\left(\frac{|a_k|}{r}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta$$

$$\left\{ \sum_{f(z_i)=0} \log \frac{r}{|z_i|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| - \log|f(0)| \right.$$

$$n(r,x) = \sum{}' \{m(s) : |\mathrm{Im}\, z| \leq r, \mathrm{Re}\, s > x\}$$

LHS 1.8

$$n(r+1,x) - n(r,x) = \sum{}' \{m(s) : r \leq |\mathrm{Im}(s)| \leq r+1, \mathrm{Re}\, s > x\}$$



$B(C_1+ir, C_2)$