Fractal Weyl laws. session 4 — Thomas Jordan

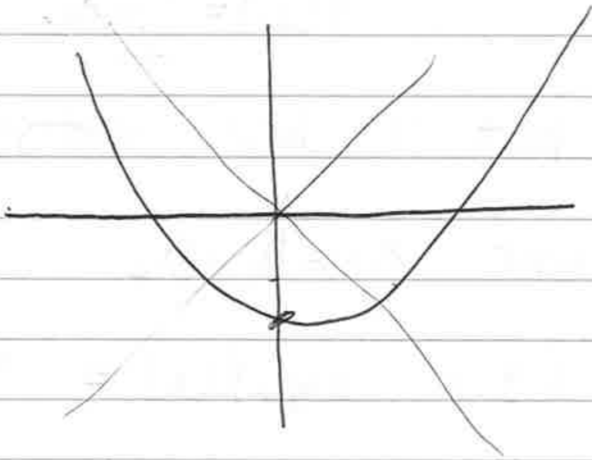
## Julia Sets for quadratic functions

$$f(z) = z^2 + c \qquad c < -2$$

The fixed points $\xi_c, -\xi_c$ $(\in \mathbb{R})$ and $|\xi_c| < c$

Take intervals

$$D_1 = [\sqrt{-c-\xi_c}, \xi_c]$$
$$D_2 = [-\xi_c, -\sqrt{-c-\xi_c}]$$

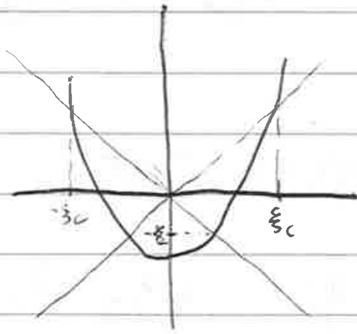


we can define

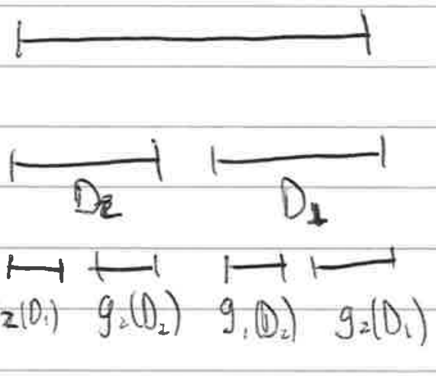
$g_1(z) = \sqrt{z-c}$,
$g_2(z) = -\sqrt{z-c}$

are such that

$$\overline{g_1(D_i)} \subseteq D_1 \qquad i=1,2$$
$$\overline{g_2(D_i)} \subseteq D_2$$




$g_1$ and $g_2$ are strict contractions on $D_i$




Iterated function system, with attractor $J$.

$$J = \bigcup_{i=1}^{2} g_i(J)$$

and it is the Julia set of $f$.

Julia set is self conformal (maps $g$'s are not linear so it is not self-similar)

Coding

$$\Pi : \Sigma = \{0,1\}^{\mathbb{N}} \to J \qquad \{1,2\}^{\mathbb{N}}$$

$$\Pi(\underline{i}) = \lim_{n\to\infty} g_{i_1} \circ \dots \circ g_{i_n}(0)$$

The point 0 doesn't matter since $g$'s are contractions.

Consider $x, y \in D_1 \cup D_2$

$$(g_{i_1} \circ \dots \circ g_{i_n})'(x) = g_{i_1}'(g_{i_2} \circ \dots \circ g_{i_n}(x)) \cdot \dots \circ g_{i_n}'(x)$$

$$\log |(g_{i_1} \circ \dots \circ g_{i_n})'(x)| - \log |(g_{i_1} \circ \dots \circ g_{i_n})'(y)| \leq C$$

(Bounded distortion, Since $g_i$ are strict contractions)

This implies that

$$\operatorname{diam}(g_{i_1} \circ \dots \circ g_{i_n}(D_1)) = (g_{i_1} \circ \dots \circ g_{i_n})'(x) \operatorname{diam}(D_1) \qquad \text{MVT}$$

Now we extend $D_1$ and $D_2$ to symmetric disks in the complex plane still with

$$\overline{g_i(D_j)} \subseteq D_i \qquad i,j = 1,2$$

Ruelle transfer operator

$$L(s)u(z) = \sum_{i=1}^{2} [g_i'(z)]^{-s} u(g_i(z)) \qquad s \in \mathbb{C}$$

acts on, $D = D_1 \cup D_2$

$H^2(D) = \{ u \text{ holomorphic in } D, \iint_D |u(z)|^2 d\mu(z) < \infty \}$

We want to define

$Z(s) = \det (I - \mathcal{L}(s))$

Let $H$ be a Hilbert space and $A : H \to H$ a compact operator, we define

$$\|A\| = \mu_0(A) \geq \mu_1(A) \dots \mu_L \to 0$$

$\mu_i$ the eigenvalues of $(A^*A)^{1/2}$

Suppose $\{\rho_j\}_{j=0}^{\infty}$ is an o.n basis of $H$, then

$$\mu_L(A) \leq \sum_{j=L}^{\infty} \|A\rho_j\|_H$$

Remember $D = D_1 \cup D_2$ $\qquad D_i$ : disks in $\mathbb{C}$ $\quad D_i = D(a_i, r_i)$

We build the following o.n. basis of $H^2(D_i)$

Take o.n basis $\{\rho_k(z)\}_{i=0}^{\infty}$

$$\rho_k(z) = \frac{\sqrt{2k+1}}{r_i} \left( \frac{z - a_i}{r} \right)^k$$

Since $\overline{g_i(D_j)} \subseteq D_i$ , then

$$\left\| \left( \frac{g_i(z) - a_i}{r_i} \right)^k \right\|_{H^2(D_i)} \leq C\alpha^k \qquad 0 < \alpha < 1$$

Define $L_{ij}(s) : H^2(D_i) \to H^2(D_j)$ by

$$L_{ij}(s)u(z) = [g_i'(z)]^s u(g_i(z)) \qquad C \text{ satisfy } |g_i'(z)|^s \leq e^{C|s|}$$

With this

$$\mu_L(L_{ij}(s)) \leq C \sum_{k \geq L} \| L_{ij}(s)\varphi_k \|$$

$$\leq C \sum_{k \geq L} e^{C|s|} \alpha^k \qquad \text{geometric}$$

$$\leq C e^{C|s|} \frac{\alpha^L}{1-\alpha}$$

$$\leq C_1 \exp(C|s| - L/C_1) \qquad \text{for suitable } C_1$$

This shows that the singular values of $L_{ij}$ decay exponentially with $L$. Since the eigenvalues of $L$ are controlled by the ~~eigenvalues~~ singular values of $L_{ij}$ $ij = 1, 2$, we will be able to show that $L$ is trace class.

Weyl inequality:
$A: H \to H$ compact, let $\lambda_i(A)$ be the eigenvalues of $A$ ~~and~~ then

$$|\lambda_0(A)| \geq |\lambda_1(A)| \geq \cdots \geq |\lambda_i(A)| \to 0$$

Then for any $N \geq 0$

$$\prod_{i=0}^{N} (1+\lambda_i(A)) \leq \prod_{i=0}^{N} (1+\mu_i(A))$$

We know $\sum_{i=0}^{\infty} \mu_i(A) < \infty$ then

$\mu$ are positive?

$$\det(I+A) = \prod_{i=1}^{\infty} (1+\lambda_i(A))$$

makes sense, and

$$\det(I+A) \leq \prod_{i=1}^{\infty} (1+\mu_i(A))$$

So

$$\det(I-L(s)) \leq \prod_{i=0}^{\infty} (1+\exp(C|s| - i/c))$$

$$\leq \exp(C^3|s|^2)$$ Esto esta raro

Prop 1:
Let $L(s): H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be the Ruelle operator. Then for all $s \in \mathbb{C}$, the operator $L(s)$ is trace class and

$$\det(I-L(s)) \leq \exp(C|s|^2)$$

$$\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\sum_{f^n(z)=z}\frac{[(f^n)']^{-s}}{1-[(f^n)'(z)]^{-1}}\right)$$

Prop 2

$$\det(I-L(s)) = \exp\left(-\sum_{n=1}^{\infty}\frac{[(f^n)'(z)]^{-s}}{1-[(f^n)'(z)]^{-1}}\right) \quad \text{for } \operatorname{Re} s \gg 0$$

Proof: $|\lambda|$ small

$$\det(I-\lambda L(s)) = \exp(\operatorname{tr}\log(I-\lambda L(s)))$$

$$= \cancel{\exp\left(-\sum_{n=1}^{\infty}\frac{\lambda^n}{n}\operatorname{tr}(L(s))^n\right)} \; \cancel{\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\sum_{f^n z=z}\right)} \; \exp\left[-\sum_{n=1}^{\infty}\frac{\lambda^n}{n}\operatorname{Tr}L^n(s)\right.$$

$$\exp\left[-\sum_{n=1}^{\infty}\frac{\lambda^n}{n}\operatorname{Tr}L^n(s)\right]$$

We can write

$$L_s u(z) = \sum_{f(w)=z}|f'(w)|^{-s}u(w)$$

Then

$$\operatorname{tr}L_s^n = \sum_{f^n z=z}\frac{|(f^{(n)})'(z)|^{-s}}{1-|(f^n)'(z)|^{-1}}$$

So

$$\det(I-\lambda L_s) = \exp\left(-\sum_{n=1}^{\infty}\frac{\lambda^n}{n}\sum_{f^n z=z}\frac{[(f^n)'(z)]^{-s}}{1-[(f^n)'(z)]^{-1}}\right)$$

Taking $\lambda=1$, we get the formula.