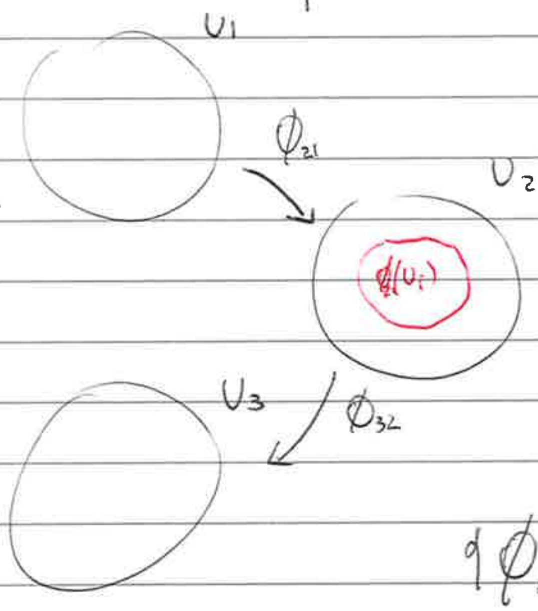
Transfer operator:



IFS

- $U_i \subseteq \mathbb{R}^d$
- $U_i = \overline{\text{int}\, U_i}$
- $A \in M_{2\times 2}(\{0,1\})$, $A^N > 0$ for some N
- $\{\phi_{ji} : U_i \to U_j : A_{ij} = 1\}$, ϕ_{ji} analytic st
- $\overline{\phi_{ji}(U_i)} \subseteq U_j$
- ϕ_{ji} is a strict contraction

$$\{\phi_{ji} : A_{ij} = 1\} = \text{IFS}$$

Composition of maps ϕ_{ij}: for a word $\underline{i} = i_1 \dots i_{n+1} \in \{1, \dots, k\}^{n+1}$ ($|\underline{i}| = n+1$) st $A_{i_k, i_{k+1}} = 1$ $k = 1, \dots, n$, we have a map

$$\phi_{\underline{i}} = \phi_{i_{n+1} i_n} \circ \dots \circ \phi_{i_2 i_1} : U_1 \to U_{n+1}$$

- **Limit set:**

$$\Lambda = \bigcap_{n=1}^{\infty} \overline{\bigcup_{|\underline{i}| = n+1} \phi_{\underline{i}}(U_{i_1})}$$

- **Fixed points:**

For $\underline{i} = i_1 \dots i_{n+1}$ st $i_1 = i_{n+1}$, we have a map

$$\phi_{\underline{i}} : U_1 \to U_1 \quad \text{contraction} \qquad (\text{Banach})$$

$\exists!$ fixed point for $\phi_{\underline{i}}$, $\phi_{\underline{i}}(z_{\underline{i}}) = z_{\underline{i}}$

$\text{Fix}_n = \{\underline{i} : i_1 = i_{n+1}\}$ $\quad$ $\widetilde{\text{Fix}}_n = \{z_{\underline{i}} : \underline{i} \in \text{Fix}_n\}$

For the system $\{\phi_{ji}: U_i \to U_j : A_{ij}\}$, we define its pressure by

$$P(s) = \lim_n \frac{1}{n} \log \sum_{\underline{i} \in Fix_n} |D\phi_{\underline{i}}(z_{\underline{i}})|^s$$

So $\dim_H \Lambda = s$ st $P(s) = 0$

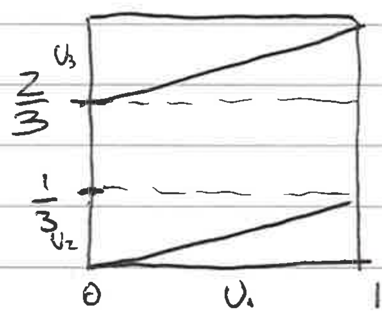
Transfer operator : $U_i \subseteq \mathbb{R}^d$

For an IFS, for a symbol $i \in \{1, \dots, k\}$, take a disk

$$D_i = D_i^{(1)} \times \dots \times D_i^{(d)} \subseteq \mathbb{C}^d \quad \text{st}$$

$$U_i \times \{0\} \subseteq D_i$$

Example



$U_i \subseteq \mathbb{R}$

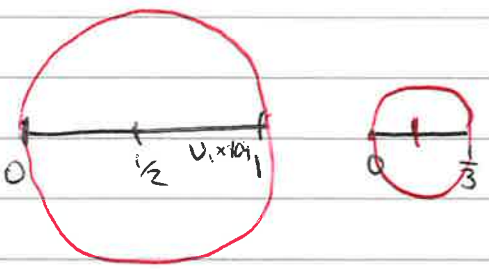
$U_1 = [0,1] \quad U_2 = [0, 1/3] \quad U_3 = [\frac{2}{3}, 1]$

$\phi_{21}: U_1 \to U_2$ $\quad$ $\phi_{31}: U_1 \to U_3$

$x \mapsto \frac{1}{3}x$ $\quad$ $x \mapsto \frac{1}{3}x + \frac{2}{3}$

$D_1 = D_1^{(1)} = \overline{D(1/2, 1/2)} \subseteq \mathbb{C}$

$D_2 = D_2^{(1)} = D(\dots$



Extend the maps from U_i to D_i,

$$\underset{U_1 \quad\quad U_2}{\phi_{21} : [0,1] \to [0, 1/3]} \qquad \Longrightarrow \qquad \phi_{21} : D_1 \longrightarrow D_2$$
$$x \longmapsto \tfrac{1}{3}x \qquad\qquad\qquad z \longmapsto \tfrac{1}{3}z$$

st
- $\overline{\phi_{ji}(D_i)} \subseteq D_j$
- $\sup\limits_{z \in D_i} |D\phi_{ji}(z)| < 1$

Define
$$D = \coprod_{i=1}^{K} D_i$$

The fixed points of $\phi_{\underline{i}} : U_{i_1} \to U_{i_1}$ are the same (and are real)

<u>Function spaces</u>

For U open, $A_\infty(U) = \{\phi : U \to \mathbb{C}$ holom, bounded in $\overline{U}\}$ with $\|\cdot\|_\infty$ is a Banach space

- Weight functions: for $s \in \mathbb{C}$, $(i,j) : A_{ij} = 1$

$$W_{s,ji} \in A_\infty(D_i) \qquad W_{s,ji}(z) = |D\phi_{ji}(z)|^s$$

- Operator $\mathcal{L}_{s,ji} : A_\infty(D_j) \to A_\infty(D_i)$

$g : D_j \to \mathbb{C}$ $\qquad (\mathcal{L}_{s,ji}\, g)(z) = g(\phi_{ji} z)\, W_{s,ji}(z)$

- ~~Component operator~~ $\cancel{\mathcal{L}_{s,i} : A_\infty\left(\coprod_{j : A_{ij}=1} D_j\right) \to A_\infty(D_i)}$

$$\cancel{\mathcal{L}_{s,i} h(z) = \sum_{j : A_{ij}=1} \mathcal{L}_{s,ji} h(z)}$$

component operator

$$\mathcal{L}_{s,i} h(z) = \sum_{j:A_{ij}=1}' h(\phi_{ji} z) W_{s,ji}(z) = \sum_{j:A_{ij}=1}' \mathcal{L}_{s,ji} h(z)$$

which is $\mathcal{L}_{s,i} : A_\infty(\coprod_{j:A_{ij}=1} D_j) \to A_\infty(D_i)$ and can be seen as an operator

$$\mathcal{L}_{s,i} : A_\infty(D) \to A_\infty(D_i) \qquad \text{for } h \in A_\infty(D),\ z \in D_i$$

$$(\mathcal{L}_{s,i} h)(z) = \mathcal{L}_{s,i} h(z).$$

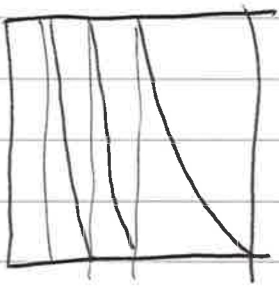
Transfer operator

$$\mathcal{L}_s : A_\infty(D) \to A_\infty(D)$$

$$(\mathcal{L}_s h)|_{D_i} = \mathcal{L}_{s,i} h \qquad h \in A_\infty(D)$$

Example: Gauss map $\quad T : [0,1) \to [0,1)$

$$x \mapsto \frac{1}{x} \bmod 1$$



$U_0 = [0,1]$

$U_n = [\frac{1}{n+1}, \frac{1}{n}]$

$$\phi_{n0} : [0,1] \to [\tfrac{1}{n+1}, \tfrac{1}{n}]$$

$$x \mapsto \frac{1}{x+n}$$

$T|_{U_n}(x) = \frac{1}{x} - n \qquad |(T^2)'(x)| \geq 4 > 1 \quad \forall x \in I$

$D = \{z : |z-1| < 3/2\}$, $\mathcal{L}_s : A_\infty(D) \to A_\infty(D)$

$$\mathcal{L}_s f(z) = \sum_{i=1}^{\infty} \left(\frac{1}{z+i}\right)^{2s} \cdot f\left(\frac{1}{z+i}\right)$$

Thm (Ruelle)
For $s \in \mathbb{R}$, $\mathcal{L}_s : A_\infty(D) \to A_\infty(D)$ has spectral radius $\exp(P(s))$, being this the unique eigenvalue of maximum modulus, and it is simple and isolated

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More about $\operatorname{spec}(\mathcal{L}_s)$:

- Nuclear op: for B Banach space, $L : B \to B$ linear bounded operator, we say L is nuclear of order p if there exist $\{u_n\} \subseteq B$, $\{l_n\} \subseteq B^*$, ($\|u_n\| = \|l_n\| = 1$) and $\{\rho_n\} \subseteq \mathbb{C}$ with $\sum_n |\rho_n|^p < \infty$ st

$$L(v) = \sum_{n=0}^{\infty} \rho_n \, l_n(v) \, u_n \qquad \forall\ v \in B$$

L is nuclear of order zero if it is nuclear of order p for every p.

Fact: L nuclear has a well defined trace

$$\operatorname{Tr}(L) = \sum_{\lambda \text{ eigenvalue}} \lambda$$

(Grothendieck)

Example: Restriction operator

Take $U \subseteq \bar{U} \subseteq V \subseteq \mathbb{C}$ Jordan domains compact

The operator

$$R_U : A_\infty(V) \to A_\infty(U)$$
$$h \mapsto R_U h = h|_U$$

is nuclear of order zero.

Reduction: $V = \{z : |z| \leq 1\}$ $U = \{z : |z| \leq r < 1\}$. For $\phi \in A_\infty(V)$, we have

$$\phi(z) = \int_{\partial V} \frac{\phi(y)}{y-z} \frac{dy}{2\pi i} = \sum_{n=1}^{\infty} \oint_{\partial V} z^{n-1} \frac{\phi(y)}{y^n} \frac{dy}{2\pi i}$$

$$= \sum_{n=1}^{\infty} \rho_n \ell_n(\phi) u_n(z)$$

where $\rho_n = r^{n-1}$, $u_n(z) = (z/r)^{n-1}$, $\ell_n(\phi) = \oint_{\partial V} \frac{\phi(y)}{y^n} \frac{dy}{2\pi i}$.

$\|u_n\|_U = 1$, $\|\ell_n\|_{A(V)} = 1$ and

$$\sum \rho_n^p = \sum_{n=1}^{\infty} r^{p(n-1)} < \infty \quad \forall \quad 0 < p \leq 1$$

so R_U is nuclear of order 0.

General case: Riemann mapping thm

Thm (Ruelle)
$\mathcal{L}_s : A_\infty(D) \to A_\infty(D)$ is nuclear of order zero.

Trace of $\mathcal{L}_s$:

Prop: $\mathrm{Tr}(\mathcal{L}_s^n) = \sum_{\underline{i} \in \mathrm{Fix}_n}' \dfrac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - D\phi_{\underline{i}}(z_{\underline{i}}))}$

Fredholm determinant is defined

$$\det(I - z\mathcal{L}_s) := \exp\left(-\sum_{n=1}^{\infty}{}' \frac{z^n}{n} \mathrm{Tr}\,\mathcal{L}_s^n\right)$$

$$= \exp\left(-\sum_{n=1}^{\infty}{}' \frac{z^n}{n} \sum_{\underline{i} \in \mathrm{Fix}_n}' \frac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - D\phi_{\underline{i}}(z_{\underline{i}}))}\right)$$

Prop (Grothendieck)
$(s, z) \mapsto \det(I - z\mathcal{L}_s)$ is entire.

If $\lambda_1(s), \lambda_2(s), \ldots$ are the eigenvalues of $\mathcal{L}_s$, then

$$\det(I - z\mathcal{L}_s) = \prod_{r=1}^{\infty} (1 - z\lambda_r(s))$$

Cor: $\dim_H \Lambda$ is the largest zero of

$$z \mapsto \det(I - z\mathcal{L}_s)$$

$\det(I - z\mathcal{L}_s)$ is entire : admits power series

$$\det(I - z\mathcal{L}_s) = 1 + \sum_{N=1}^{\infty}{}' \, d_N(s) z^N$$

Prop:

$$d_N(s) = \sum_{\substack{(n_1,\dots,n_m) \\ n_1+\dots+n_m=N}}{}' \frac{(-1)^m}{m!} \prod_{l=1}^{m} \frac{1}{n_l} \sum_{\underline{i} \in \mathrm{Fix}_n}{}' \frac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - D\phi_{\underline{i}}(z_{\underline{i}}))}$$

There exist $0<\delta<1$ st $d_N(s) = O(\delta^{N^{1+1/d}})$ as $N\to\infty$ $\forall s>0$

Julia set

For $U \subseteq \mathbb{C}$ a domain, $f: U \to U$ holomorphic

$$J = \overline{\bigcup_{n \geq 1} \{ z \in \mathbb{C} : f^n z = z \text{ and } |(f^n)'(z)| > 1 \}}$$

J is closed and invariant by f

Lemma: (Ruelle, Bowen)
If $f: J \to J$ is expanding, there exists a Markov partition associated to the system.

- Cuadratic maps: $f_c(z) = z^2 + c$, J_c
- Mandelbrot set $M = \{ c \in \mathbb{C} : |f_c^n(0)| \not\to \infty \text{ as } n \to \infty \}$

Prop:
For f_c cuadratic with Julia set J_c, $f_c: J_c \to J_c$ is hyperbolic iff $c \notin M$ or f_c has an attracting periodic point ($f_c^n z = z$, $|(f^n)'(z)| < 1$).

(hyperbolic: no eigenvalues with $|\ | = 1$)