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Session 3: Thermodynamic formalism

Pressure

M compact metric space, $f: M \to M$, $\phi: M \to \mathbb{R}$ continuous.

$$P(f,\phi) = \lim_{\varepsilon \to 0} \limsup_{n} \frac{1}{n} \log \sup_{E\ (n,\varepsilon)\text{-gen}} \left(\sum_{x \in E} \exp \sum_{k=0}^{n-1} \phi \circ f^k(x) \right)$$

E is (n,ε)-generating if $M \subset \bigcup_{a \in E} B(a,n,\varepsilon)$ where

$B(a,n,\varepsilon) = \{x \in M : d(f^i(x), f^i(a)) < \varepsilon \quad i=0,\dots,n-1\}$

Variational Principle:

$$P(f,\phi) = \sup_{\mu \in \mathcal{M}} \{ h_\mu(f) + \int_M \phi \, d\mu \}$$

If μ attains the sup, we call it an equilibrium measure

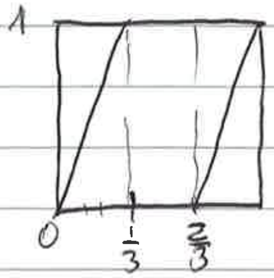
Properties

Regard $P(f,\cdot)$ as a function on $C^0(M,\mathbb{R})$ with $\|\cdot\|_\infty$ norm, then

1. $P(f,\cdot)$ is 1-Lipschitz $\quad |P(f,\phi) - P(f,\psi)| \le \|\phi - \psi\|_\infty$
2. $P(f,\phi + c) = P(f,\phi) + c \quad \forall c \in \mathbb{R}$
3. If $\phi \le \psi \Rightarrow P(f,\phi) \le P(f,\psi)$
4. $P(f,\cdot)$ is convex.
5. $P(f,\phi) = P(f, \phi + u \circ f - u) \quad \forall \ u \in C^0(M,\mathbb{R})$

Example

Consider



$$f : [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1] \to [0,1]$$
$$x \longmapsto 3x \bmod 1$$

Then f^2 is defined on

and so on.

Define $\Lambda = \bigcap_{k \geq 1} f^{-k}([0,\tfrac{1}{3}] \cup [\tfrac{2}{3}, 1])$ cantor set

$f^n : \Lambda \to \Lambda$ is well defined $\forall n \geq 0$.

$f : \Lambda \to \Lambda$ is conjugated to $\sigma : \Sigma^+ \to \Sigma^+ = \{0,1\}^{\mathbb{N}}$

$\phi = -t \log|f'| = -t \log|3|$ and solve $P(f, \phi) = 0$:

$$P(f, \phi) = P(f, 0 - t\log|3|) = P(f, 0) - t\log 3$$
$$= h(f) - t\log 3 = \log 2 - t \log 3$$

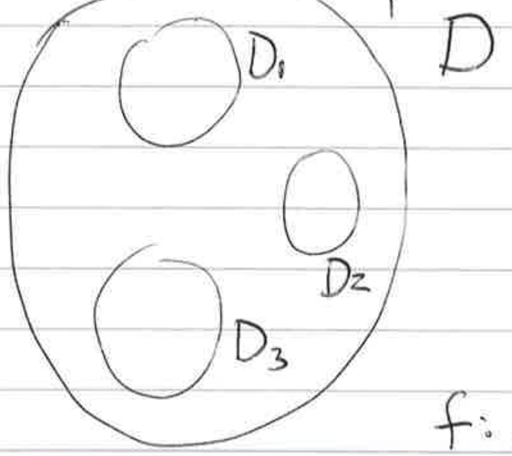
so $t = \dfrac{\log 2}{\log 3} = \dim_H \Lambda$

Thm

Let $f : M \to M$ topologically ~~mixing~~ exact expansive transformation and $\phi : M \to \mathbb{R}$ Hölder. Then

$$P(f, \phi) = \lim_n \frac{1}{n} \log \sum_{x \in \mathrm{Fix}(f^n)} \exp\left(\sum_{k=0}^{n-1} \phi \circ f^k(x)\right)$$

Bowen's Equation



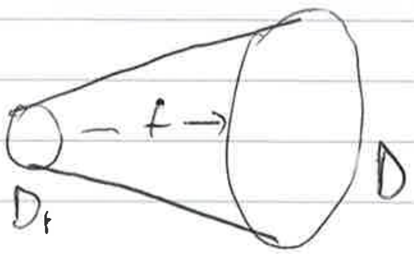
D, D_i compact convex $\subseteq \mathbb{R}^n$

$D_i \subseteq D$

$D_i \cap D_j = \emptyset \quad i \neq j$

$$\mathrm{vol}(D \setminus \underbrace{\cup D_i}_{= D^*}) > 0.$$

$f : D^* \to D$ $\quad C^1$ st



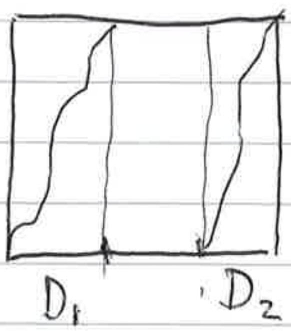
$f|_{D_i} \to D$ is a homeo. Set

$$\Lambda = \bigcap_{k \geq 1} f^{-k}(D^*)$$

- f is ~~expansive~~ expanding, $\|Df'\| \geq \beta > 1$ $\Big\}$ $\|D_f f^n v\| \geq c\beta^n \|v\|$ $\quad \beta > 1$
- $\log|Df|$ is Hölder,
- f is conformal, Df is a multiple of an isom at each point. Then, if $s = \dim_H \Lambda$,

$$P(s) := P_\Lambda(f, -s\log|Df|) = 0$$

We give an sketch of proof in the case $n = 1$, and $D^* = D_1 \cup D_2$.



Step 1: There is a unique sol to $P(s)=0$.

If $t_1 \le t_2 \Rightarrow -t_2 \log|f'| \le -t_1 \log|f'|$

$$\Rightarrow P(t_2) \le P(t_1)$$

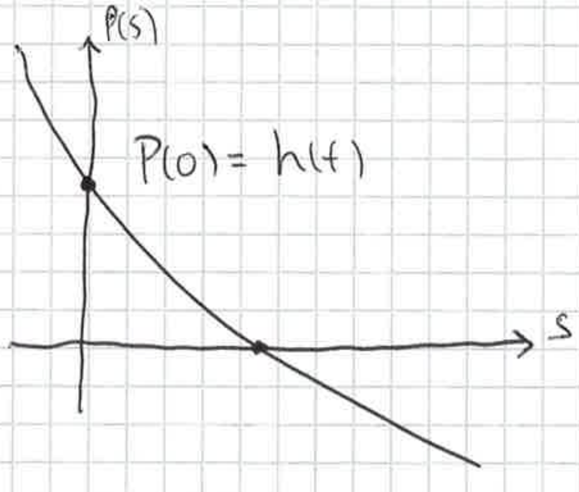
so $P(t)$ is decreasing.

Note that

$$P(t) = \sup_{\mu} \{ h_\mu(f) + \int -t\log|f'| \, d\mu \}$$

$$\le \log 2 - t \log \beta \longrightarrow -\infty \quad \text{as} \quad t \to \infty$$

So by IVT there exists a solution to $P(s)=0$.



We set some notation: for $(w_1, \dots, w_n) \in \{1,2\}^n$, set

$$D_{w_1 \dots w_n} = \bigcap_{k=1}^{n} f^{-k}(D_{w_k})$$ so we have a map

$$f^n : D_{w_1 \dots w_n} \longrightarrow D$$

Which is a homeomorphism. Since f is C^1 and D compact, there is $0 < c < 1$ so that

$$\operatorname{diam} D_{w_1 \cdots w_n} \leq c^n$$

(This follows from the MVT)

Step 2: Bounded distortion:

$$\operatorname{diam} D_{w_1 \cdots w_n} \asymp |(f^n)'(x)|^{-1}, \quad x \in D_{w_1 \cdots w_n}$$

that is, $\exists\, B_1, B_2 > 0$ st

$$B_1 \leq \frac{\operatorname{diam} D_{w_1 \cdots w_n}}{|(f^n)'(x)|^{-1}} \leq B_2 \quad \forall\, w \in \Sigma,\ x \in D_{w_1 \cdots w_n}$$

Proof:
Since $\log|f'|$ is Hölder

$$\big|\log|f'(x)| - \log|f'(y)|\big| \leq a|x-y|^\alpha \quad \text{so}$$

$$\begin{aligned}\big|\log|f'(f^k(x))| - \log|f'(f^k(y))|\big| &\leq a|f^k x - f^k y| \\ &\leq a(\operatorname{diam} D_{w_{k+1} \cdots w_n})^\alpha \\ &\leq a\, c^{\alpha(n-k)}\end{aligned}$$

for $x, y \in D_{w_1 \cdots w_n}$. Now

$$\left| \sum_{k=0}^{n-1} \log|f'(f^k(x))| - \sum_{k=0}^{n-1} \log|f'(f^k(y))| \right|$$

$$\leq \sum_{k=0}^{n-1} \left| \operatorname{Log}|f'(f^k x)| - \operatorname{Log}|f'(f^k y)| \right|$$

$$\leq \sum_{k=0}^{n-1} a\, C^{\alpha(n-1-k)} < \infty$$

This is equiv to

$$M_1 \leq \left| \frac{(f^n)'(x)}{(f^n)'(y)} \right| \leq M_2 \quad , x,y \in D_{w_1 \dots w_n}$$

By the MVT, $\exists\ B_1, B_2 > 0$ st

$$B_1 \leq \frac{\operatorname{diam} D_{w_1 \dots w_n}}{|(f^n)'(x)|^{-1}} \leq B_2 \qquad x \in D_{w_1 \dots w_n}$$

Remark:
There exists $M > 0$ st

$$M \operatorname{diam} D_{w_1 \dots w_n} \leq \operatorname{diam} D_{w_1 \dots w_{n+1}}$$

$$\begin{aligned} \operatorname{diam} D_{w_1 \dots w_{n+1}} \geq B_1 |(f^{n+1})'(x)|^{-1} &\geq B_1 |(f^n)'(x)|^{-1} \cdot |f'(f^n(x))|^{-1} \\ &\geq M\, B_2\, |(f^n)'(x)|^{-1} \\ &\geq M \operatorname{diam} D_{w_1 \dots w_n} \end{aligned}$$

$\square$

Step 3: Gibbs measures

A measure μ_t is called Gibbs (Borel, supported in Λ) for $-t\log|f'|$ if $\exists\ c > 0$ st

$$\frac{1}{C} \leq \frac{\mu_t(D_{w_1 \dots w_n})}{|(f^n)'(x)|^{-t}} \leq C \qquad \forall\ x \in D_{w_1 \dots w_n}$$

Lemma: there exists a unique Gibbs measure $\forall\, t$

$$\mu_n = \frac{1}{S_n} \sum_{f^n x = x} \exp \sum_{k=0}^{n-1} -t\log|f' \circ f^k x|\ \delta_x$$

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~~Step~~ Proof of the thm:

- We have a unique sol t^* of $P(s)=0$;
- Let $\mu = \mu_{t^*}$ the Gibbs measure for $-t^*\log|f'|$, ie,

$$\frac{1}{C} \leq \frac{\mu(D_{w_1 \cdots w_n})}{|(f^n)'(x)|^{-t^*}} \leq C \qquad x \in D_{w_1 \cdots w_n}$$

so $$\frac{1}{C'} \leq \frac{\mu(D_{w_1 \cdots w_n})}{(\operatorname{diam} D_{w_1 \cdots w_n})^{-t^*}} \leq C'$$

- Let $r>0$ small and take $D_{w_1 \cdots w_n}$ st

$$\operatorname{diam} D_{w_1 \cdots w_n} \leq r < \operatorname{diam} D_{w_1 \cdots w_{n-1}} \leq M^{-1} \operatorname{diam} D_{w_1 \cdots w_n}$$

$$\operatorname{diam} D_{w_1 \cdots w_n} \leq r < M^{-1} \operatorname{diam} D_{w_1 \cdots w_n}$$

There exists $\lambda > 0$ st

$$\Lambda \cap B(x, \lambda r) \subseteq D_{w_1 \cdots w_n} \subseteq \Lambda \cap B(x, r)$$

In fact

$$\operatorname{diam} D_{w_1 \cdots w_n} \leq r \Rightarrow D_{w_1 \cdots w_n} \subseteq \Lambda \cap B(x, r)$$

~~on the other side~~

$$\cancel{2rM^2 < 2M \operatorname{diam} D_{w_1 \cdots w_n} \leq 2 \operatorname{diam} D_{w_1 \cdots w_{n+1}}}$$

$$\cancel{\lambda r < 2 \operatorname{diam} D_{w_1 \cdots w_n}}$$

$$\cancel{\lambda = 2M^2}$$

3D: $d = \mathrm{dist}(D_1, D_2)$

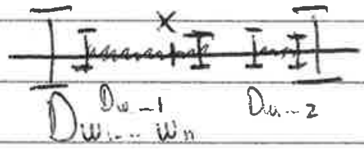
$$\mathrm{diam}\, D_{w_1 \dots w_n} \le \mathrm{dist}(D_{w_1 \dots w_n 1}, D_{w_1 \dots w_n 2}) \le \mathrm{diam}\, D_{w_1 \dots w_n}$$

~~So~~ So if $\mathrm{diam}\, D_{w_1 \dots w_n} \le r < M^{-1} \mathrm{diam}\, D_{w_1 \dots w_n}$

$\Rightarrow$ $Mr < \mathrm{diam}\, D_{w_1 \dots w_n}$

$M^2 r < \mathrm{diam}\, D_{w_1 \dots w_{n+1}}$ $\quad$ $MEr < E\, \mathrm{diam}_{w_1 \dots w_n} \le \mathrm{dist}$

So $\lambda = \min\{M^2, ME\}$ $\quad$ $\lambda r < \mathrm{diam}\, D_{w_1 \dots w_{n+1}}$



So $\mu(B(x,r)) \le \mu(\cancel{B}(D_{w_1 \dots w_n})) \le \mu(B(x,r))$

$$\frac{1}{C} \mu(B(x,r)) \le \mathrm{diam}(D_{w_1 \dots w_n})^{t^*} \le C \mu(B(x,r))$$

$$\tilde{C} r^{t^*} \le \mathrm{diam}(D_{w_1 \dots w_n})^{t^*} \le \hat{C} r^{t^*}$$

By the mass distribution principle,

$$\tilde{C} \le H^{t^*}(\Lambda) \le \hat{C}$$

$\Rightarrow$ $\dim_H \Lambda = t^*$