Fractal Weyl Laws. session 2 Sebastian Muller

Non-fractal Weyl laws

Eigenvalues of the Hamiltonian (self-adjoint)

$$\hat{H} = \frac{\hat{\underline{p}}^2}{2m} + U(\underline{r}) \qquad \hat{p} = \frac{\hbar}{i}\nabla$$

$$= -\frac{\hbar^2}{2m}\Delta + U(\underline{r}) \qquad 2m := 1$$

acting on $L^2_{loc}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$

e.g. billiards, 2d, Ω has finite area, $U=0$, Dirichlet boundary conditions

$$\hat{H}\psi_d = E_d\psi_d \qquad d = 1, 2, \ldots$$

$$\underset{\mathbb{R}}{\wedge}$$

ψ_d are normalized. If ψ depends on t, the above arises if one solves

$$\hat{H}\psi(r,t) = i\hbar\frac{\partial}{\partial t}\psi(r,t)$$

by separation.

Classical dynamics

$$\hat{\underline{p}} \mapsto \underline{p}$$

$$\hat{H} \mapsto H(\underline{r}, \underline{p})$$

The Hamilton eq

$$\dot{\underline{r}} = \frac{\partial H}{\partial \underline{p}} \qquad \dot{\underline{p}} = -\frac{\partial H}{\partial \underline{r}}$$

Counting eigenvalues:

counting function $N(E) = \#\{E_d \leq E\} = \sum_d \underset{\text{Heaviside}}{\Theta}(E - E_d)$

Level density: $d(E) = N'(E) = \sum_{d} \delta(E-E_d)$

Weyl Law

$$N(E) = \frac{1}{(2\pi\hbar)^n} \int d^n r \int d^n p \; \Theta(E - H(\underline{r},\underline{p}))$$

+ terms of lower order in E

$$d(E) = \frac{1}{(2\pi\hbar)^n} \int d^n r \int d^n p \; \delta(E - H(\underline{r},\underline{p})) + \text{terms of} \ldots$$

this is not true, but

$\langle d(E) \rangle_E$: average over window in E, or restrict to smooth test function

For billiards

$$N(E) = \frac{m|\Omega| E}{2\pi\hbar^2} + \mathcal{O}(E)$$

$$\langle d(E) \rangle_E = \frac{m|\Omega|}{2\pi\hbar^2} + \mathcal{O}(1)$$

Derivation (physicsy)

Time evolution operator $e^{-\frac{i}{\hbar}\hat{H}t}$

$$\psi(\underline{r},t) = \exp(-\tfrac{i}{\hbar}\hat{H}t)\,\psi(\underline{r},0)$$

Propagator: integral kernel of $\exp(-\frac{i}{\hbar}\hat{H}t)$:

$$K(\underline{r},\underline{r}_0,t) = \exp(\tfrac{-i}{\hbar}\hat{H}t)\,\delta(\underline{r}-\underline{r}_0)$$

$$= \sum_j \psi_j(r) \exp(\tfrac{-i}{\hbar}E_j t)\,\overline{\psi_j(\underline{r}_0)}$$

$$G(\underline{r},\underline{r}_0,E^+) = \frac{-i}{\hbar}\int_0^\infty dt \, \exp(iE^+t/\hbar)\,K(\underline{r},\underline{r}_0,t)$$

$$= \sum_j \frac{\psi_j(\underline{r})\,\overline{\psi_j(\underline{r}_0)}}{E^+ - E_j}$$

energy dependent Green function, which is the integral kernel of the resolvent

$$\frac{1}{E^+ - \hat{H}}$$

Level density:

$$d(E) = -\frac{1}{\pi} \lim_{\eta\to 0} \mathrm{Im} \int d^2r \, G(\underline{r}, \underline{r}, E^+)$$

as RHS $= \sum_j \lim_{\eta \to 0} \frac{1}{\pi} \frac{\eta}{(E-E_j)^2+\eta^2} = \sum_j \delta(E - E_j)$ (convergence in the sense of distributions)

Short-time propagator

$$K(\underline{r}, \underline{r_0}, t) = \exp\left(-\frac{i}{\hbar}\hat{H}t\right) \underbrace{\delta(\underline{r}-\underline{r_0})}_{\frac{1}{(2\pi\hbar)^n}\int d^np \exp\left(\frac{i}{\hbar}\underline{p}\cdot(\underline{r}-\underline{r_0})\right)}$$

$$= \underbrace{\frac{1}{(2\pi\hbar)^n} \int d^np \, \exp\left[\frac{i}{\hbar}\left(\underline{p}\cdot(\underline{r}-\underline{r_0}) - H(\underline{r_0}, \underline{p})t\right)\right]}_{K_0(\underline{r}, \underline{r_0}, t)} + O(t^2)$$

$$\mathrm{Im}\, G_0(\underline{r}, \underline{r}, E^+) = \mathrm{Im} -\frac{i}{\hbar} \int_0^\infty \exp(iE^+t/\hbar) \, K_0(\underline{r}, \underline{r}, t)$$

$$= \mathrm{Im} \frac{1}{(2\pi\hbar)^n} \int d^np \frac{1}{E^+ - H(\underline{r}, \underline{p})}$$

$$d_0(E) = \frac{1}{(2\pi\hbar)^n} \int d^nr \int d^np \, \delta(E - H(\underline{r}, \underline{p}))$$

Trace formula

e.g. dispersing billiards 

$$d(E) \underset{E\to\infty}{\sim} \frac{m|\Omega|}{2\pi\hbar^2} + \frac{m}{\pi\hbar^2 k} \sum_p{}' A_p \cos(k \text{ Length of the orbit}) \qquad E = \frac{\hbar^2 k^2}{2m}$$

$$h u(x) = \sum_{|\alpha|\le n}{}' a_\alpha(x) D^\alpha u(x) \qquad \alpha = (\alpha_1, \dots, \alpha_d) \quad |\alpha| = \sum_{j=1}^{d} \alpha_j$$

symbol

$$\sigma(x,\xi) = \sum_{r,s=1}^{d}{}' a_{rs}(x)\xi_r\xi_s + \sum_{b=1}^{d}{}' i a_b(x)\xi_b + a(x)$$

$[a_{rs}(x)]_{r,s=1}^{d}$ σ is elliptic if this matrix is real symmetric and has positive eigenvalues, $x\in\Omega$ and are uniformly bounded by pos const.

Laplacian: $\Delta u(x) = \sum_{j=1}^{d}{}' \frac{\partial^2}{\partial x_j^2} u(x) \qquad x \in \Omega$

Sobolev-Space:

Let $\ell \in \mathbb{N}$, the ℓ-th derivative sobolev space is the Hilbert space of $L^2(\Omega)$ whose ℓ-th derivative is also $L^2(\Omega)$

$$H^\ell(\Omega) = \{ u \in L^2(\Omega) : \int_\Omega \sum_{|\alpha|=\ell}{}' |D^\alpha u(x)|^2 dx < \infty \}$$

$$\langle u, v \rangle_{H^\ell} = \int_\Omega u(x)\overline{v(x)} + \sum_{|\alpha|=\ell}{}' D^\alpha u(x) \overline{D^\alpha v(x)}\, dx$$

For L an 2^{nd} order elliptic D.O., let

$$D_0 := \{ u \in H_0^1(\Omega) \mid Lu \in L^2(\Omega) \}$$

Then L with domain D_0 defines an adjoint operator in $L^2(\Omega)$.

$H^{\ell}_0(\Omega)$: completion of $L^{\infty}_0(\Omega)$, the smooth functions supported ~~on~~ by Ω.

Let

$$q_N(u,v) = \int_\Omega dx\, \nabla u(x)\overline{\nabla v(x)} \qquad u,v \in H^1(\Omega).$$

assume ~~$\partial\Omega$~~ $\partial\Omega$ smooth, let $\tilde{D}$ - set of all smooth functions $u \in C^\infty(\Omega)$ st

$$\frac{\partial u}{\partial n}(x)\Big|_{\partial\Omega} := n(x)\nabla u(x)\Big|_{\partial\Omega} = 0$$

Closure of q_N in $\tilde{D}$ is $q_N\ H^1$

Let $-\Delta_N$ self adjoint Neumann Laplacian

Variational Techniques

Rayleigh quotient: A self adjoint, $u \in \mathrm{Dom}(q_A)$

$$R(u) = \frac{q_A(u,u)}{\langle u,u\rangle} \quad \left(= \frac{\langle Au, u\rangle}{\langle u,u\rangle} \text{ if } u \qquad \right)$$

3×3 Hermitian matrix, $\lambda_1 \le \lambda_2 \le \lambda_3$, u_1, u_2, u_3 Let $u = \sum \alpha_j u_j$

$$R(u) = \frac{\sum_j \lambda_j |\alpha_j|^2}{\sum_j |\alpha_j|^2}$$

$\lambda_1 = \min\{R(u) \mid u \in \mathbb{C}^3\}$

$\lambda_2 = \min\{R(u) \mid u \perp \text{span}(u_1)\}$

$\lambda_3 = \min\{R(u) \mid u \perp \text{span}(u_1, u_2)\}$

λ_j are stationary points of the map $R: \mathbb{C}^3 \to \mathbb{R}$.

Consider $S \subseteq \mathbb{C}^3$, $\exists\, \tilde{u} \in S \setminus \{0\}$ s.t. $\tilde{u} \perp u_1$, $R(\tilde{u}) \geq \lambda_2$
so that

$$\max_{u \in S} R(u) \geq \lambda_2 \quad \text{and}$$

$$\lambda_2 = \min_{\dim S = 2} \max_{u \in S} R(u)$$

$$\lambda_3 = \min_{\dim S = 3} \max_{u \in S} R(u)$$

Thm (Min-Max principle)
A self adjoint st $R(u) \geq C$ $\forall u \in \text{Dom}(A)$, $C \geq 0$. Let D be either $\text{Dom}(A)$ or $\text{Dom}(q_A)$, and ~~let~~ then

$$-C \leq \mu_1 \leq \mu_2$$

$$\mu_k = \min_{\substack{\dim(S) = k \\ S \subseteq D}} \max_{u \in S} R(u)$$

- If $\dim(H) < \infty$, then $\text{spec}\, A = \{\mu_k\}$ counting w/ multiplicity
- If $\dim(H) = \infty$, ~~then~~ $E := \lim_k \mu_k$, then

$$E - \min(\text{spec}_{ess} A), \quad \text{spec}_{dis}(A) \cap (-\infty, t) = \{\mu_k\}$$

Denote by $v_j(A)$ the components of A discrete spectrum
Write

$$\text{spec}\, A \leq \text{spec}\, B \quad \text{if} \quad v_j(A) \leq v_j(B) \quad \forall j$$

Denote by $-\Delta_D(\Omega)$ $\cancel{(-\Delta_N)}$ $(-\Delta_N(\Omega))$ the Dirichlet (Neuman) laplacian on bounded set Ω, open

- Dirichlet eigenvalues: $\lambda_j = \lambda_j(\Omega) := v_j(-\Delta_D(\Omega))$
- Neuman $\mu_j = \mu_j(\Omega) = v_j(-\Delta_N(\Omega))$

Domain monotonicity:

Let $\Omega' \subseteq \Omega$ bounded open in $\mathbb{R}^d$. Then $\forall j \geq 1$

$$\lambda_j(\Omega) \leq \lambda_j(\Omega')$$

Idea of proof:

Min-max characterization and $H_0^1(\Omega') \subseteq H_0^1(\Omega)$

$v \in H_0^1(\Omega')$

$$u(x) = \begin{cases} v(x) & x \in \Omega \\ 0 & x \in \Omega \setminus \Omega' \end{cases} \in H_0^1(\Omega)$$

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Dirichlet-Neuman Bracket

Monotonicity of eigenvalues of quadratic form w/r to

Thm:

$\Omega \subseteq \mathbb{R}^d$ open, $\partial\Omega$ sufficiently small, then

$$\mu_j(\Omega) \leq \lambda_j(\Omega) \quad \forall j$$

This follows from $H_0^1(\Omega) \subseteq H^1(\Omega)$.

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Counting function for L discrete spectrum

$$N(\lambda; L) = \#\{v_j \in \mathrm{spec}(L) \mid v_j \leq \lambda\}$$

Let Q_a square of side a, let $-\Delta_D(Q_a)$, can show that

$$\operatorname{spec}(-\Delta_D(Q_a)) = \left\{ \frac{\pi^2}{a^2}(k^2+m^2) \;\middle|\; (k,m)\in\mathbb{Z}^2_+ \right\}$$

For $\lambda_{k,m} \leq \lambda \Rightarrow k^2+m^2 \leq \frac{a^2\lambda}{\pi^2}$ so $N(\lambda, -\Delta_D(Q_a)) = \#$ of integer lattice points inside the 1-st quadrant of the circle of radius $\frac{a\sqrt{\lambda}}{\pi}$.

<u>Gauss</u>: $\#\{(k,m)\in\mathbb{Z}^2 \mid |(k,m)| < R\} = \pi R^2 + o(R^2)$, $R\to\infty$

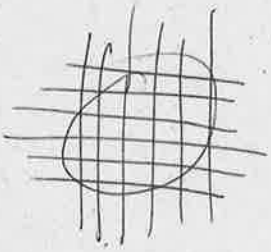
so
$$N(\lambda; -\Delta_D(Q_a)) = \frac{a^2\lambda}{4\pi} + o(\lambda)$$
$$= \frac{1}{4\pi}|Q_a|_2\lambda + o(\lambda) \qquad |\Omega|_d:\ \text{d-dim volumen}$$

<u>Thm:</u>
$\Omega \subseteq \mathbb{R}^d$ bounded, $\partial\Omega$ suf regular, then with $-\Delta(\Omega)$ either Dirichlet or Neuman

$$N(\lambda; -\Delta(\Omega)) = (2\pi)^{-d}\omega_+|\Omega|_d\lambda^{d/2} + o(\lambda^{d/2})$$

$\omega_t := (\pi)^{d/2}/\Gamma(1+\frac{d}{2})$: vol unit ball in $\mathbb{R}^d$

<u>Proof</u> $(d=2)$



For $\Omega\subseteq\mathbb{R}^2$, choose $a\geq 0$ small and consider $\underset{\text{squares}}{Q_{a_i}}$ $i\in\mathbb{N}$

$I = \{Q_{a_i}$ inside $\Omega\}$ $B = \{Q_{a_i}$ intersect $\partial\Omega\}$

so by Dirichlet monotonicity and DN bracket

$$N(\lambda, -\Delta_D(\Omega)) \geq N(\lambda, -\Delta_D(\bigcup_I Q_{a_i})) \geq \sum_i N(\lambda; -\Delta_D(Q_{a_i})) - \sum \frac{1}{(4\pi)}|Q_{a_i}|_2\lambda + o(\lambda)$$
$$\geq \frac{1}{4\pi}(|\Omega|_2 - \varepsilon)\lambda + o(\lambda)$$

on the other side,

$$N(\lambda; -\Delta_D(\Omega)) \leq N(\lambda; -\Delta_D(\bigcup_{i\in I\cup B} Q_{a_i})) \leq \sum_{i\in I\cup B} N(\lambda; -\Delta_N(Q_{a_i}))$$
$$\sum_{i\in I\cup B}\frac{1}{(4\pi)}|Q_{a_i}|_2\lambda + o(\lambda) \leq \frac{1}{4\pi}((|\Omega|_2+\varepsilon)\lambda + o(\lambda)$$

Taking $\varepsilon\to 0$ $(a\to 0)$,

$$N(\lambda, \Delta\Omega)) = \frac{1}{4\pi}|\Omega|_2\lambda + o(\lambda) \quad \blacksquare$$