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Fractal Weyl laws, Session 1

Shifts

Alphabet: $A=\{1,\dots,m\}$

Word: finite sequence of elements of A

$\Sigma_m = A^{\mathbb{Z}}$

$\Sigma_m^+ = A^{\mathbb{N}}$

The (left) shift operator

$$\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}} \qquad \sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$$
$$(\sigma x)_i = x_{i+1} \qquad (\sigma x)_i = x_{i+1}$$

Full two-sided shift: (Σ_m, σ)

one-sided (Σ_m^+, σ)

Metric for shifts: Σ_m, Σ_m^+ are compact in the product topology. This topology can be induced by a metric. The product topology has a basis of cylinders

$$C^{n_1 \dots n_k}_{j_1 \dots j_k} = \{(x_\ell) : x_{n_j} = j_i\} \text{ for } n_1 < n_2 \dots < n_k, \ j_i \in A_m$$

σ^{-1}(cylinder) is a cylinder, so σ is discrete.

A metric inducing this topology

$$d(x,x') = 2^{-\ell} \qquad \ell = \min\{|i| : x_i \neq x_i'\}$$

Subshifts

X is a closed shift invariant. (X,σ) is a subshift. Define the adjacency matrix $A=(A_{ij}) \in M(n\times n, \{0,1\})$.

$$\Sigma_A = \{(x_i) \in \Sigma_m : A_{x_i, x_{i+1}} = 1\}$$

$$\Sigma_A^+ = \{(x_i) \in \Sigma_m^+ \qquad \}$$

Are the subshifts of finite type.

A is transitive if there exists $N > 0$ st $A^N > 0$. (A non-negative)

Fact:

A is transitive $\Rightarrow (\Sigma_A, \sigma)$ is topologically mixing (for every U, V open sets $\exists M > 0$ st $\sigma^{-t}(U) \cap V \neq \emptyset$ for $t \geq M$).

Markov measures

$P \in M(n \times n)$ is called stochastic if

- $P \in M(m \times m, \mathbb{R}_+)$
- $\sum_{j=1}^{m} P_{ij} = 1$ for all i

This matrix is compatible with A if $P_{ij} > 0 \Leftrightarrow A_{ij} = 1$

Thm (Perron-Frobenius)

B transitive matrix, non-negative entries has a unique eigenvector $\vec{v}$ with all positive coordinates. The eigenvalue corresponding to $\vec{v}$ is positive, simple, and with the largest absolute value.

Cor:

$P\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$

So 1 is the max eigenvalue, and hence P^t has 1 as max eigenvalue.

Let

$\vec{p}^{\,t} = \begin{pmatrix}p_1\\ \vdots\\ p_m\end{pmatrix}$ st $\sum_{i=1}^{m} p_i = 1$ and $P^t \vec{p}^{\,t} = \vec{p}^{\,t}$ (vectors are rows)

Transposing, we get $pP = p$ so

$$\sum_{i=1}^{m} p_i P_{ij} = p_j$$

Define a measure μ on cylinders

$$\mu\left(C^{i, i+1, \dots, i+k}_{j_0, \dots, j_k}\right) = p_{j_0} P_{j_0 j_1} P_{j_1 j_2} \cdot \dots \cdot P_{j_{k-1} j_k}$$

By Kolmogorov thm, μ is a probability measure on the whole σ-algebra. This prob is also shift invariant

Example: (Bernoulli measures)

Pick a probability ~~measure~~ vector $\vec{p} = (p_1 \dots p_m)$

$$\mu_{\vec{p}}\left(C^{i, i+1, \dots, i+k}_{j_0, \dots, j_k}\right) = \prod_{l=0}^{k} p_{j_l}$$

The corresponding matrix P is
$$P = \begin{pmatrix} p_1 & \dots & p_m \\ p_1 & \dots & p_m \\ \vdots & & \vdots \\ p_1 & \dots & p_m \end{pmatrix}$$

Hausdorff dimension

Hausdorff measure (work in $\mathbb{R}^n$)

For $X \subset \mathbb{R}^n$ and $\delta > 0$, $\{U_i\}_{i=1}^{\infty}$ is a δ-cover of X if

- $X \subseteq \bigcup_{i=1}^{\infty} U_i$
- $|U_i| \leq \delta$

Let $s > 0$, $\delta > 0$, define

$$\mathcal{H}^s_\delta(X) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-cover of } X\right\}$$

$$\mathcal{H}^s(X) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(X)$$

$\mathcal{H}^s$ is the s-dimensional Hausdorff measure of X.

Thm

Let $s > 0$, then $\mathcal{H}^s$ is an outer measure on $\mathbb{R}^n$, and the $\mathcal{H}^s$-mesurable sets include the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^n}$

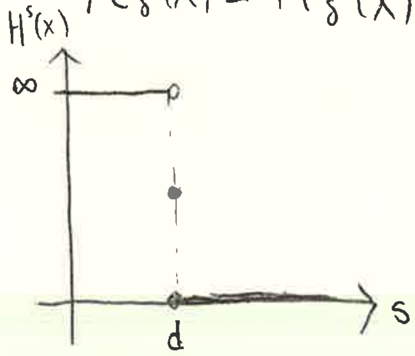
Properties:

- $\mathcal{H}^s(\lambda X) = \lambda^s \mathcal{H}^s(X)$ for $\lambda > 0$

- For $t > s > 0$,

$$\sum_{i=0}^{\infty} |U_i|^t \le \delta^{t-s} \sum_{i=1}^{\infty} |U_i|^s \qquad \{U_i\} \text{ a } \delta\text{-cover}$$

so $\mathcal{H}^t_\delta(X) \le \mathcal{H}^{t-s}_\delta(X)$ and hence we get a graph of $\mathcal{H}^s(X)$



the point $d \in [0, \infty]$

We call

$$d = \dim_H(X) = \inf\{s \ge 0 : \mathcal{H}^s(X) = 0\}$$

the Hausdorff dimension

Properties:

1) $\dim_H U = n$ for every non-empty open set $U \subseteq \mathbb{R}^n$
2) $E \subset F \Rightarrow \dim_H(E) \le \dim_H(F)$
3) $\{A_i\}_{i=1}^{\infty} \Rightarrow \dim_H \bigcup_{i=1}^{\infty} A_i = \sup_{i=1,\dots} \dim_H A_i$
4) For $f: X \to X$, bi-Lipschitz, then $\dim_H X = \dim_H f(X)$

Thm: (Mass distribution principle)
For $X \subseteq \mathbb{R}^n$, μ a finite measure, $\mu(X) > 0$. Then if there exists $s \ge 0$, $C > 0$, $\delta_0 > 0$ st

$$\mu(U) \le C|U|^s$$

for all non-empty ~~open~~ sets $U \subseteq \mathbb{R}^n$ with $\operatorname{diam} U \le \delta_0$. Then $\mathcal{H}^s(X) \ge \frac{\mu(X)}{C}$ and hence, $\dim_H X \ge s$.

Thm (Frostman's lemma)
Let $s>0$, $B \subseteq \mathbb{R}^n$ a Borel set. Then $\mathcal{H}^s(B)>0$ iff there exists a Borel measure μ st $\mu(B)>0$ and s.t.

$$\mu(B(x,r)) \le r^s \qquad \text{for all } x\in\mathbb{R}^n,\ r>0.$$

Thermodynamic Formalism

- Topological entropy

Let (X,d) be a compact metric space, $f: X\to X$ continuous. Define a metric

$$d_n(x,y) = \max_{0\le k\le n-1} d(f^k x, f^k y)$$

$$\mathrm{cov}(n,\varepsilon,f) = \text{minimum cardinality of a cover of } X \text{ by sets with } d_n\text{-diameter} < \varepsilon. \quad < \infty$$

by compactness

A subset A is (n,ε)-separated if any 2 distinct points in A are at least ε apart in the d_n-metric.

$$\mathrm{sep}(n,\varepsilon,f) = \text{maximum cardinality of an } (n,\varepsilon)\text{-separated set} \quad < \infty$$

Now, the topological entropy is

$$h(f) = \lim_{\varepsilon\to0^+}\lim_{n\to\infty}\frac{1}{n}\log \mathrm{cov}(n,\varepsilon,f)$$

$$= \lim_{\varepsilon\to0^+}\overline{\lim_{n\to\infty}}\frac{1}{n}\log \mathrm{sep}(n,\varepsilon,f) = \lim_{\varepsilon\to0^+}\underline{\lim_{n\to\infty}}\frac{1}{n}\log \mathrm{sep}(n,\varepsilon,f)$$

- Measure theoretic entropy

Let $(X,\mathbb{B},\mu,T)$ a compact space X, $\mathbb{B}$ a Borel σ-alg and T a μ-invariant transformation. A collection $\{A_i\}_{i=1}^N \subseteq \mathbb{B}$ is a finite partition if $X = \coprod_{i=1}^N A_i$. For two partitions $P_1=\{A_1\ldots A_n\}$, $P_2=\{C_1,\ldots,C_m\}$ Their Joint is

$$P_1\vee P_2 = \{A_i\cap C_j : i=1\ldots n,\ j=1\ldots m\}$$

The entropy of a partition P_1 is

$$H(P_1) = -\sum_{i=1}^{n} \mu(A_i) \log \mu(A_i)$$

Consider

$$\bigvee_{j=0}^{n-1} T^{-j} P_1 = \{ \bigcap_{i=0}^{n-1} T^{-i} A_{ij} : A_{ij} \in P_1 , j \in \{1, \dots n\} \}$$

$$h(T, P_1) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} T^{-j} P_1\right)$$

is the entropy of (T,μ) w/r to P_1. The measure theoretic entropy of (T,μ) is

$$h_\mu(T) = \sup\{h(T,P) : P \text{ finite partition of } X\}$$

- Topological pressure

Let (X,d) a metric space, $f: X \to X$ continuous, $\varphi : X \to \mathbb{R}$ continuous

$$P_n(f,\varphi,\varepsilon) = \sup \{ \sum_{x\in E} \exp\left(\sum_{i=0}^{n-1} (\varphi \circ f^i)(x)\right) : E \text{ is an } (n,\varepsilon) \text{ separated set of } X\}$$

The topological pressure of φ for f

$$P(f,\varphi) = \lim_{\varepsilon\to 0^+} \overline{\lim_{n\to\infty}} \frac{1}{n} \log P_n(f,\varphi,\varepsilon)$$

Remark

If $\varphi = 0$, then $P(f,0) = h(f)$.

Variational principle:

Thm:

$$P(f,\varphi) = \sup\{ h_\mu(f) + \int_X \varphi \, d\mu : \mu \text{ is an } f\text{-inv prob measure} \}$$

$$h(f) = \sup \{ h_\mu(f) : \mu \text{ f inv prob}\}$$