My research

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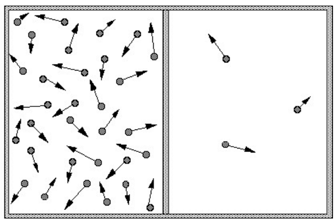
May 12, 2018

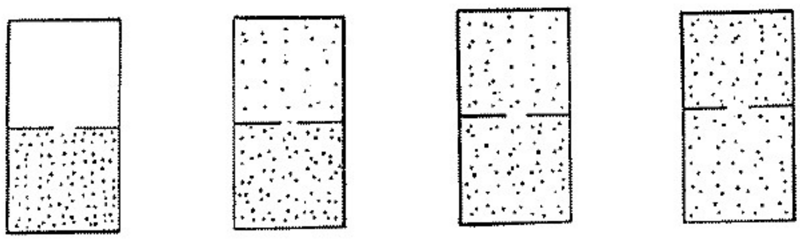
In my research, I study certain class of dynamical systems with highly chaotic behaviors. This investigation lies on the interface of Fractal Geometry, Probability Theory and Dynamical Systems.

The theory of dynamical systems models abstractly systems that evolve in time according to certain rules. These rules can be deterministic or have some degree of uncertainty, although the theory is more developed for the first case kind of systems. In this theory, one of the aims is to describe the complexity of a given system. Some systems exhibit what is called a *chaotic behavior*; this is, despite of their evolution being described by well defined deterministic rules, uncertainty in the measurements makes predictions impossible in the long timescale.

Consider two closed boxes of the same size, containing both a certain amount of the same gas. Suppose that the first box contains 100 times more of that gas than the second one (see figure). It is apparent that the first box represents a more complex system than the second one, and we can make this more precise introducing the concept of entropy.

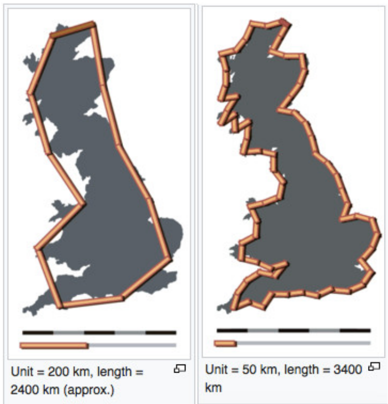


As stated by Boltzmann in his famous formula, we can think of the entropy of a system as the logarithm total number of configurations accessible to the system. This takes into account different factors, such as number of particles, degrees of freedom of each particle, and intrinsic properties of the gas particles (such as internal degrees of freedom). Thus, a system with ten times more particles would have twice as much entropy. Physicists of the 19th and 20th centuries realized that systems such as gases, even if treated classically (that is, ignoring the quantum effects), are impossible to handle via the deterministic approach of Newton's equations. In practice, this would imply to solve systems of around 10^{23} coupled differential equations, task which is practically impossible even for the most powerful computers nowadays. The revolutionary insight in this regard, was the introduction of statistical methods in the theory of thermodynamics. Questions such as "what is the velocity/momentum of a given particle" became less important, given rise to questions such as "what is the velocity/momentum of the *average* particle of the system". In this setting, Boltzmann formulated the following question (loosely speaking): if we start with a gas using half the space of an empty box and we let the system evolve in time according to the laws of motion, will the particles of the gas equidistribute in the box (see figure below)?



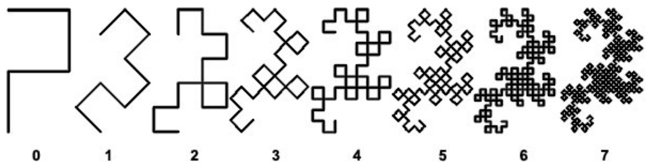
Boltzmann was not sure about the answer of this question, and he assumed it to be positive. This is know as the *ergodic hypothesis*. The previous question is again, formulated in statistical terms, since equidistribution is precisely the asymptotic equivalence between time and spaces averages of occupation times. More precisely, if we fix a region A of the box (let us say, the right half of it), and we follow the trajectory of a given particle as time goes, if we count the number of visits that our fixed particles does to the subregion A and divide it by the total amount of time spent, we get the proportion of time spent in the region A. If we let the time go to infinity, and the result of this proportion only depends on the size of the region, we say that the trajectory of the particle is equidistributed in the box. If this happens, we can conclude that the particle spends the same proportion of time in every region of the same size, and so it goes around the whole box as time goes. Ergodic theory arose as a mathematical abstraction of the ideas behind statistical mechanics. The cornerstone result of ergodic theory, Birkhoff's ergodic theorem is indeed an equidistribution theorem for abstract dynamical systems.

Fractal geometry arises as a systematic way to study objects which seem far more *rough* than the idealized Euclidian models that have been studied for thousands of years. Mandelbrot, in is paper "How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension", he observed that if we try to measure the length of the coastline of Britain, we would get different results as we use different scales of measurements. In fact, as we make the scale of the measurements smaller, the length of the coastline grows arbitrarily large (see figure below).

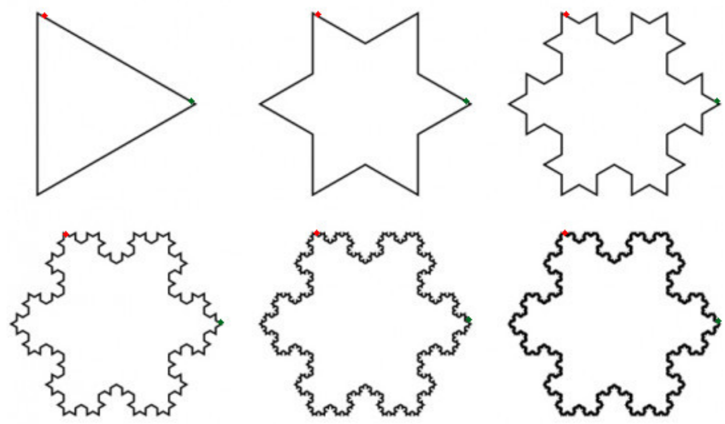


In the same paper, Mandelbrot suggested that the coastline of Britain is not a one dimensional object such as a straight line or a circumference, but an object of *fractional dimension*. That is, the coastline is something *in between* a line and a surface. He also observed that this kind of objects tend to exhibit *self-similarity* properties, that is, if we look at them at different scales (analogously to zooming in), they have roughly the same shape. Some objects in nature have been suggested to have fractal properties (see figure below).

Some fractals are constructed using dynamical processes, establishing a direct connection between the theories of Dynamical Systems and Ergodic theory, and Fractal Geometry (see figure below).



Suppose now we construct a fractal using an iterative method, and we follow the orbit of two different points as we iterate the process (red and green points in the figure below). Then, the distance between these points increases arbitrarily as we iterate the process. The same situation occurs for points which are close enough.



The *Lyapunov exponent* measures how fast does the distance between two given points increases as we iterate the process defining the fractal. Thus, the largest the Lyapunov exponent, the more chaotic the system is. This notion of complexity is in principle of a different nature than the entropy, but for a large portion of fractals, they are closely related. In fact, under some regularity conditions, we have that

$$\text{dimension of the fractal} = \frac{\text{entropy}}{\text{Lyapunov exponent}}.$$

In my research, I study what happens with this relation when the iterative processes involved are highly chaotic, this is, both measures of complexity, the entropy and the Lyapunov exponent are equal to infinity, and so the above formula would give nothing but an indeterminacy of the form ∞/∞.

Keywords— Dynamical systems, Ergodic theory, Probability theory, Fractal geometry.