Sizing Small Sets

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When do we say a set $S \subset \mathbb{R}^n$ is small?
Different approaches:

- Set theory (cardinality),
- Measure theory (measure),
- Topology (meagre),
- Category theory (small set),
- Dimension theory (dimension).

The Cantor set:

$$\mathcal{C} = \{x \in [0,1] \text{ such that there are no 1 on its ternary expansion}\}$$

The Cantor set... kind of.

How small is the Cantor set?

- It is uncountable,
- It has zero measure,
- It is Meagre,
- dimension = ... ?

Sensible definition of dimension: we would like it to have some reasonable properties

- Manifolds have the dimension we expect they to have,
- Finite and countable sets should have dimension zero,
- Open sets of $\mathbb{R}^n$ have dimension n,
- If $E \subset F$, then $\dim E \leq \dim F$.
- If $\{E_i\}$ is a countable collection of sets, then

$$\dim \left(\bigcup_i E_i \right) = \sup_i \dim E_i.$$

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There are different notions of dimension,

- Algebraic (Krull dimension): very restrictive (varieties) and only takes integer values.
- Topological: easy to define, not restrictive, but only takes integer values. Can be very hard to compute.
- Vector space dimension: very restrictive, only takes integer values. Very easy to compute.
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Hausdorff measures: for $s \geq 0$ and $A \subset \mathbb{R}^n$,

$$H^s(A) = \lim_{\delta \to 0} \inf\{\sum_{i=1}^{\infty} (\texttt{diam } U_i)^s : A \subseteq \bigcup_{i=1}^{\infty} U_i, \ \texttt{diam } U_i < \delta\},$$

which means...

Properties of the Hausdorff measures:

1. $H^s(\cdot)$ are outer measures, so they can measure everything.
2. They scale nicely: $H^s(\lambda \cdot A) = \lambda^s H^s(A)$.
3. If $E \subset F$, then $H^s(E) \leq H^s(F)$.
4. $H^s(\cdot)$ is invariant under isometries.
5. For integer values of s, H^s corresponds to the s−dimensional Lebesgue measure (up to normalization).
6. If $H^s(A) < \infty$ then $H^t(A) = 0$ for $t > s$ and $H^r(A) = \infty$ for $r < s$.

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The number s from the last property is the right parameter to measure the set A with. We define then

$$\dim_H(A) = \inf\{s \geq 0 : H^s(A) = 0\} = \sup\{s \geq 0 : H^s(A) = \infty\}.$$

So what is the Hausdorff dimension of the Cantor set?

Other examples: Let $x \in [0,1]$, and denote by $a_n(x)$ the $n-$th digit of its binary expansion. The frequency of apparition of the digits 0 and 1 in the binary expansion of x is

$$f_0(x) = \lim_{n\to\infty} \frac{1}{n} \#\{i \in \{1,\dots,n : a_i(x) = 0\}\},$$
$$f_1(x) = \lim_{n\to\infty} \frac{1}{n} \#\{i \in \{1,\dots,n : a_i(x) = 1\}\}.$$

The Birkhoff's ergodic theorems tells us (among many things) that

$$\mathtt{Leb}\{x \in [0,1] : f_0(x) = f_1(x) = 1/2\} = 1.$$

So the set of points having odd frequencies of digits are *small*. How small?

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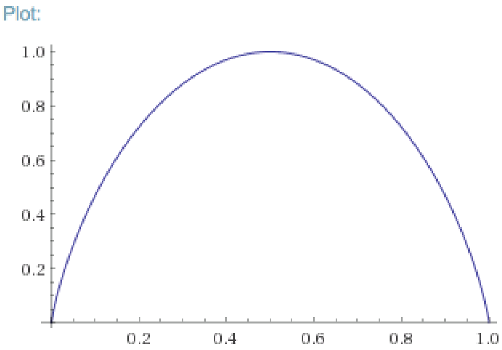
Theorem (Besicovitch).

The Hausdorff dimension of the sets $J_\alpha = \{x \in [0,1] : f_0(x) = \alpha, f_1(x) = 1-\alpha\}$ *is given by*

$$\dim_H J_\alpha = \frac{-1}{\log 2}(\alpha \log \alpha + (1-\alpha)\log(1-\alpha)).$$

The plot of the function $\alpha \mapsto \dim_H J_\alpha$:



Looks nice...

Another example:

Theorem (Kintchin).

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a non-increasing function. Then the set

$$\left\{x \in [0,1] : \left|x - \frac{p}{q}\right| < \frac{\psi(q)}{|q|} \text{ for i.m. } \frac{p}{q} \in \mathbb{Q}\right\}$$

has Lebesgue measure 1 *if* $\sum_q \psi(q) = \infty$ *and Lebesgue measure* 0 *if* $\sum_q \psi(q) < \infty$.

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An immediate consequence of Kintchine's theorem is that the sets

$$E_c = \Big\{ x \in [0,1] : \Big| x - \frac{p}{q} \Big| < \frac{1}{|q|^{1+c}} \text{ for i.m. } \quad \frac{p}{q} \in \mathbb{Q} \Big\}$$

have zero Lebesgue measure for $c > 1$, so E_c is indeed a small set from the point of view of Lebesgue measure. How small is E_c?

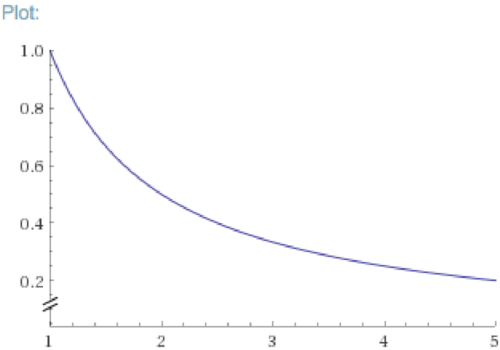
Jarnik's theorem gives us the answer:

Theorem (Jarnik).

The Hausdorff dimension of the sets E_c is given by $\dim_H E_c = 1/c$ for $c > 1$.

In particular, the Lioville numbers have zero dimension (despite being uncountable!).

The plot of the function $c \mapsto \dim_H E_c$:



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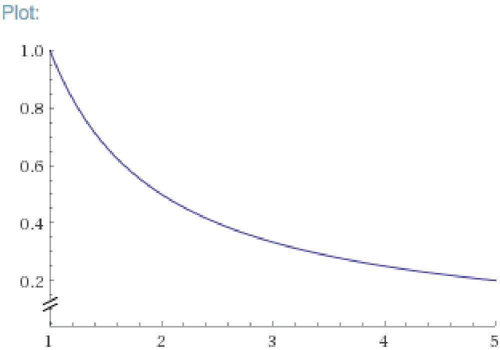

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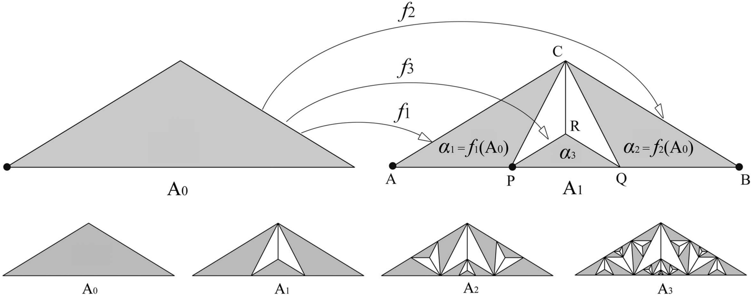
Here is when dynamics come into play: iterated function systems (IFS).



Dynamical systems and ergodic theory are to fractal geometry what linear algebra is to plane geometry.

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The ingredients of an IFS:

1. A closed bounded interval $I \subset \mathbb{R}$ and $I_1, \ldots, I_m$ pairwise disjoint closed subintervals of I.
2. A set $\{f_i : I \to I_i\}_{i=1}^m$ of strict (sufficiently smooth) contractions.

With this, we have a fractal!

$$\Lambda = \bigcup_{i=1}^{m} f(\Lambda).$$

Explicitly, if $\Omega = \{1, \ldots, m\}^{\mathbb{N}}$, then

$$\Lambda = \{x = \lim_{k \to \infty} (f_{i_k} \circ \ldots \circ f_{i_1})(x_0), \ \{i_n\} \in \Omega\}$$

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What is the Hausdorff dimension of Λ?

Consider the operator $T_s \colon \mathcal{C}(I) \to \mathcal{C}(I)$ given by

$$(T_s)(h)(x) = \sum_{j=1}^{m} h(f_j(x)) \log |f_j'(f_j(x))|^s.$$

Under certain reasonable hypothesis on the IFS, this operator has spectral gap.

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Theorem (Bowen, Ruelle).

Let λ_s be the maximal eigenvalue of T_s. Then the Hausdorff dimension of Λ is the unique solution s_0 to the equation

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The perturbation theory of operators asserts that if we have an operator $T^{(0)}$ acting with spectral gap (with maximal eigenvalue λ_0) on a reasonably good Banach space, then any sufficiently small analytic perturbation $T^{(t)}$ also has spectral gap (with maximal eigenvalue λ_t) and moreover, the function

$$t \mapsto \lambda_t$$

is analytic.

So Hausdorff dimension may vary analytically!

(Yet) another consequence of the Birkhoff's ergodic theorem:

Theorem.

For almost every $x \in [0,1]$, there is a sequence of rational numbers $\{p_n/q_n\}$ such that

$$\lambda(x) := -\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n}{q_n} \right| = \frac{\pi^2}{6\log 2},$$

i.e., $|x - \frac{p_n}{q_n}| \asymp \exp(-n\lambda(x))$. Here $\{p_n/q_n\}$ is the best rational approximation of its class of complexity.

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What is the size of the sets having a different speed of approximation?

Theorem (Policott & Weiss, Kessebomer & Stratman).
Let

$$J(\beta) = \{x : \lambda(x) = \beta\},$$

then the function $\tilde{t} : \beta \mapsto \dim_H J(\beta)$ *has domain* $\left(2\log\left(\frac{1+\sqrt{5}}{2}\right), \infty\right)$*, it is real analytic and attains its unique maximum at* $x = \frac{\pi^2}{6\log 2}$*. For every such* β*, the set* $J(\beta)$ *has positive dimension and it is dense.*

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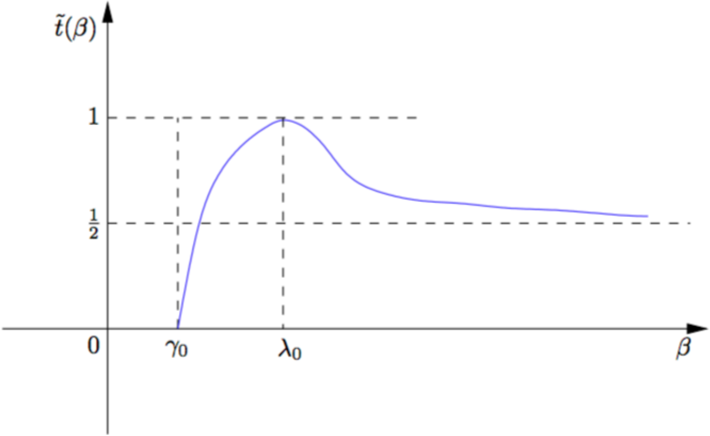
The function $\tilde{t}$ coding the dimension of the sets $J(\beta)$ was described in more details by Fan, Liao, Wang and Wu:



This shows a very intricate decomposition of the set $[0,1]$, know as *multifractal decomposition*.

The real question now is...

What is the dimension of

Romanesco Broccoli

According to this paper,

Fractal dimensions of a green broccoli and a white cauliflower

Sang-Hoon Kim*

Division of Liberal Arts, Mokpo National Maritime University,
Mokpo 530-729, and *Institute for Condensed Matter Theory,*
Chonnam National University, Gwangju 500-757, Korea

(Dated: February 2, 2008)

$$\dim(\texttt{Green broccoli}) \approx 2.7$$
$$\dim(\texttt{White cauliflower}) \approx 2.8$$

Thanks