Infinite entropy and zero dimensional measures

FELIPE PÉREZ
School of Mathematics, University of Bristol
fp16987@bristol.ac.uk

Setting: infinite entropy

Classical result: for a *nice* map T of the unit interval I:

$$\boxed{\dim_H \mu = \frac{h(\mu)}{\lambda(\mu)},}$$

where $h(\mu)$ and $\lambda(\mu)$ represent the entropy and the Lyapunov exponent of μ, when they are both finite.

Question: what happens when $h(\mu) = \lambda(\mu) = \infty$?

Our model: Gauss-like maps and Bernoulli measures (see figure, left). This **includes** the Gauss map.

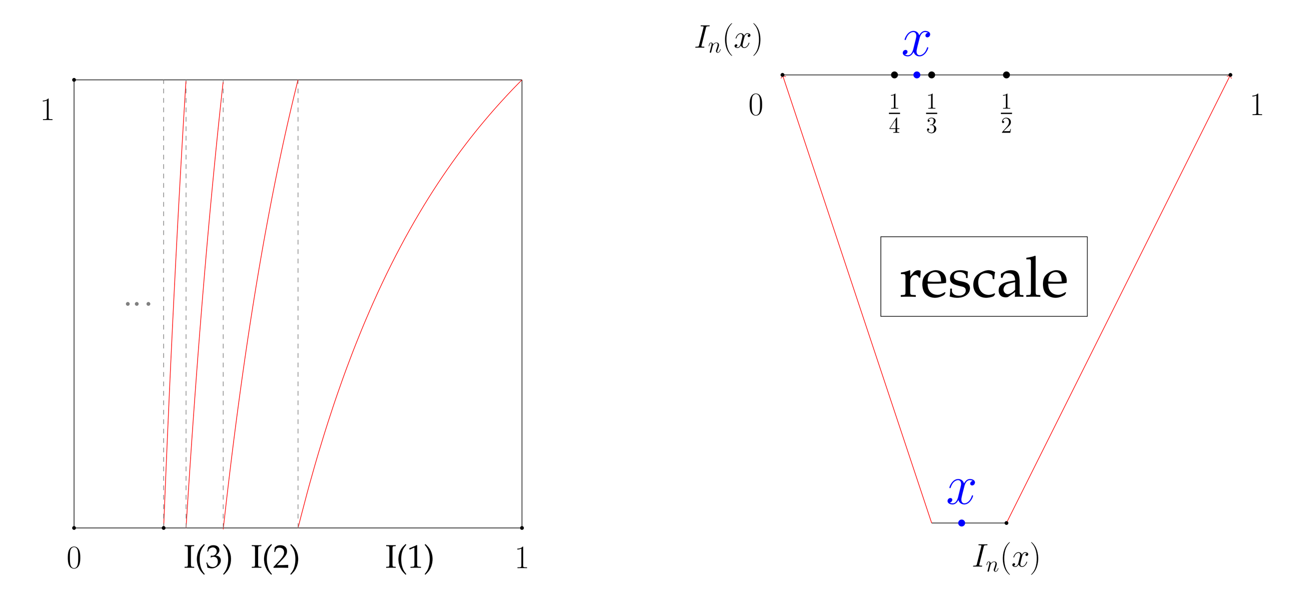

Figure: (left) a Gauss-like system; (right) the coding process for a generic point. In each step, we rescale I_n and see where does x lie.

We make use of the Markov partition structure of the system

$$I(k_1, \ldots, k_n) = I_{k_1} \cap T^{-1}(I_{k_2}) \cap \ldots \cap T^{-(n-1)}(I_{k_n}).$$

Set $r_k = |I(k)|$ and $p_k = \mu(I(k))$.

Assumptions:

1. Bounded distortion $+\varepsilon$,
2. Polynomial decay of $|I(k)|$ given by $1/\alpha = s_\infty := \inf\{t \geq 0 : \sum_{n\geq 1}^{\infty} r_n^t < \infty\}$
3. The limit

$$s := \lim_{n\to\infty} \frac{\log p_n}{\log r_n}.$$

exists.

For our model, the entropy and Lyapunov exponent are

$$h(\mu) = -\sum_{n=1}^{\infty} p_n \log p_n \ , \ \lambda(\mu) = -\sum_{n=1}^{\infty} p_n \log r_n.$$

We assume $h = \lambda = \infty$.

Markov dimensional exactness

Markov dimension:

$$\delta(x) := \lim_{n\to\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|}$$

We compute this for our measures:

Theorem 1. *The Markov dimension of μ exists and it is equal to $\delta(x) = s$, at μ almost every point.*

Local dimension: not exact

The lower/upper local dimensions are given by:

$$\underline{d}(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \ , \ \overline{d}(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

These two limits are equal under very mild conditions in the finite Lyapunov exponent case. For infinite entropy systems, this is not the case:

Theorem 2. *The lower dimension satisfies $\underline{d}(x) \leq \hat{s}$ μ-almost everywhere. For Gauss-like systems, $\hat{s} = 0$, and hence $\dim_H \mu = 0$.*

Theorem 3. *For μ almost every x, we have the following inequalities:*

$$1/\alpha = s_\infty \leq s = \overline{d}(x).$$

This implies that $0 = \dim_H \mu < 1/\alpha = \dim_P \mu$.

Some words about the proof

- $h(\mu) = \lambda(\mu) = \infty$ implies that $-\log p_{k_1(x)}$ and $-\log r_{k_1(x)}$ are not integrable,
- This makes Birkhoff's averages be very *wild*,
- Some ideas of infinite ergodic theory are needed,
- Appropriate covers to detect the asymptotic interaction between p_n and r_n.

Extensions and questions

Extensions: These results can be extended to a certain class of Gibbs measures, but such class is harder to characterize, as more sophisticated tools are required.
Questions:

1. Is there a measure for which $0 < \dim_H \mu$? or $\dim_H \mu = 1$?
2. What happens in the general case, where no independence properties are assumed?
3. Is there an ergodic approach to the general case? (e.g., via the suspension flow).