Session 3

May 12, 2018

1 Bowen's formula

Recall that in the first session, we introduced the topological pressure for dynamic on compact spaces. More precisely,

Definition. *Let $f : M \to M$ a continuous function in a compact metric space M and $\phi : M \to \mathbb{R}$ a continuous function. The topological pressure of ϕ with respect to f is*

$$P(f,\phi) = \lim_{\epsilon \to 0} \limsup_{n} \frac{1}{n} \log \sup \left\{ \sum_{x \in E} \exp \left(\sum_{k=0}^{n-1} (\phi \circ f^k)(x) \right) : E \text{ is an } (n,\epsilon)\text{-generating set for } M \right\}$$

The topological pressure satisfies the variational principle, which relates it with the measure theoretic entropy. In fact,

Theorem 1.1 (Variational Principle)**.** *Let $f : M \to M$ be a continuous transformation on a compact metric space. Then, for every continuous function $\phi : M \to \mathbb{R}$, we have that*

$$P(f,\phi) = \sup \left\{ h_\mu(f) + \int_M \phi \, d\mu \right\},$$

where the supremum is taken over all $f-$ invariant probability measures μ on M.

A measure attaining the supremum is called an *equilibrium measure.*

Now we present some easy properties of the topological pressure that allows us to calculate it in some cases:

Theorem 1.2. *Regard $P(f,\cdot)$ as a function defined on $C^0(M,\mathbb{R})$ with the supremum norm, then*

1. *$P(f,\cdot)$ is Lipschitz continuous, with Lipschitz constant equal to 1;*
2. *$P(f,\phi + c) = P(f,\phi) + c$ for every $c \in \mathbb{R}$;*
3. *if $\phi \leq \psi$ then $P(f,\phi) \leq P(f,\psi)$;*
4. *$P(f,\cdot)$ is convex, that is, $P(f, t\phi + (1-t)\psi) \leq tP(f,\phi) + (1-t)P(f,\psi)$;*
5. *$P(f,\cdot)$ is constant in every cohomology class, that is, $P(f,\phi) = P(f,\phi + u \circ f - u)$ for every $u \in C^0(X,\mathbb{R})$;*

We use this to calculate the topological pressure of a particular system: do the cantor set example as the repeller of the iterated function system. Take $\phi_t = -t\log|T'|$, calculate it, calculate $P(f,\phi_0)$ and calculate d such that $P(f,\phi_d) = 0$. Plot the function $P(f,\phi_t)$

This phenomena is a particular case of a more general, which we will explain later. Before going in depth with this, we state a characterization of the pressure for nice systems.

For full shifts and other systems, topological pressure can be characterized as the weighted (by ϕ) exponential growth of periodic points of the dynamic f. In fact,

Definition. *A function $f : M \to M$ is said to be* topologically exact *if for every open set $U \subset M$, there exists $N \in \mathbb{N}$ such that $f^N(U) = M$.*

Theorem 1.3. *Let $f : M \to M$ a topologically exact expansive transformation and $\phi : M \to \mathbb{R}$ a Hölder potential. Then,*

$$P(f,\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\in \mathrm{Fix}(f^n)} \exp\left(\sum_{k=0}^{n-1} (\phi \circ f^k)(x)\right).$$

Now we state one of the most important results connecting thermodynamic formalism and dimension theory.

Theorem 1.4 (Bowen's equation)**.** *Let $D, D_1, ..., D_N \subset \mathbb{R}^d$ compact convex sets such that $D_i \subset D$ and $D_i \cap D_j$ for $i \neq j$. Set $D_* = D_1 \cup ... \cup D_N$ and suppose that*

$$vol(D \setminus D_*) > 0.$$

Suppose there exists a $\mathcal{C}^1$ function $f : D_* \to D$ such that the restriction to every D_i is an homeomorphism. Set

$$\Lambda = \bigcap_{k=0}^{\infty} f^{-k}(D_*).$$

We make the following hypothesis for f:

1. *f is expansive in D_*,*
2. *$\log|Df|$ is Hölder in D_*,*
3. *f is conformal, this is, $\|Df(x)\|\|Df(x)^{-1}\| = 1$ for every $x \in D_*$*

The, the Hausdorff Dimension of Λ is sd, where s is the unique solution of the equation

$$P(s) := P(-s\log|\det Df|) = 0.$$

Sketch of proof. We present a sketch of proof for the case $d = 1$ and $N = 2$. We need a series of lemmas to prove this.

Lemma 1.5. *There exists a unique solution s to the equation $P(-t\log|f'|) = 0$.*

Proof. In fact, it is possible to see that the system is conjugated to a full shift on 2 symbols, so it has topological entropy equal to $\log 2 > 0$, and this is precisely the value of the left hand side of the equation at $t = 0$. On the other side, by the Variational Principle,

$$\begin{aligned} P(-t\log|f'|) &= \sup\{h_\mu(f) + \int(-t\log|f'|)d\mu\} \\ &\leq \log 2 - t\sup\{\log|f'(x)| : x \in \Lambda\}, \end{aligned}$$

so letting $t \to \infty$, we get that $P(-t\log|f'|) \to -\infty$. By the Intermediate Value Theorem, we conclude that there exists a root of the equation $P(-t\log|f'|) = 0$. The uniqueness follows from the fact that P is monotonous, so $P(-t\log|f'|)$ is decreasing in t. We call this unique root by t_0. □

Recall the mass distribution principle:

Lemma 1.6 (Mass distribution Principle)**.** *Let μ be a probability measure on a compact metric space M and suppose there exist numbers $d, K, r > 0$ such that*

$$\mu(B) \leq K(\operatorname{diam} B)^d \tag{1}$$

for every measurable set $B \subset M$ with $\operatorname{diam} B < r$. *Then if $\mu(A) > 0$ we have $m(A, d) > 0$ and hence* $\dim_H A \geq d$.

Our system (Λ, f) can be coded by a full shift $(\Sigma = \{1, 2\}^{\mathbb{N}}, \sigma)$ where the coding $\pi : \Sigma \to \Lambda$ is given by

$$\pi(\omega) = \bigcap_{k=0}^{\infty} f^{-k} D_{\omega_k}$$

and in a diagram,

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \Lambda & \xrightarrow{f} & \Lambda \end{array}$$

$\star$ it seems that this is irrelevant $\star$

A distortion property, implied by the regularity assumptions on f is the following:

Lemma 1.7 (Bounded distortion)**.** *There exists constants $B_1, B_2 > 0$ such that*

$$B_1 \leq \frac{\operatorname{diam} D_{i_1,\ldots,i_n}}{|(f^n)'(x)|^{-1}} \leq B_2$$

for every $x \in D_{i_1,\ldots,i_n}$, $n \in \mathbb{N}$ and $(i_1, \ldots) \in \Sigma$.

We introduce now the notion of Gibbs measure:

Definition. *Suppose μ is a σ-invariant probability measure in Σ_A^+ and $\phi : \Sigma_A^+ \to \mathbb{R}^+$ a continuous function. Then μ is called a* Gibbs Measure *if there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \leq \frac{\mu(C_{i_1,\ldots,i_n})}{\exp(-nP(\phi) + \sum_{k=0}^{n-1}\phi(\sigma^k\omega))} \leq C_2$$

for every $n \in \mathbb{N}$ and $\omega \in C_{i_1,\ldots,i_n}$.

Applying this to $\phi = -s \log |f' \circ \pi^{-1}|$, we obtain that $\mu(C_{i_1,\ldots,i_n}) \asymp |(f^n)(x)|^{-s}$ for $x \in C_{i_1,\ldots,i_n}$

Lemma 1.8. *There exists a Gibbs measure associated to $\phi(x) = -s \log |f'(\pi^{-1}(x))|$ where s is the solution to the pressure equation. This measure is the unique equilibrium measure for phi.*

Proof. Construct measures supported in the periodic points of $f : \Lambda \to \Lambda$ by

$$\mu_n = \frac{1}{s_n} \sum_{f^n x = x} \exp(\sum_{k=0}^{n-1} \phi \circ f^k(x)) \delta_x$$

where s_n are normalization constants in order to obtain $\mu(\Lambda) = 1$ and δ_x the Dirac measure supported in $\{x\}$. Since the space of probability measures is compact, there exists an accumulation point μ of the sequence $\{\mu_n\}$. This measures satisfied the Gibbs property. □

Now we proceed to prove the Bowen's formula. Let s be the unique solution to the pressure equation, and μ the Gibbs measure associated to $-s \log |f'(\pi^{-1}(x))|$. □

2 Julia sets

Definition. *Let $U \subseteq \mathbb{C}$ a domain and $f : U \to U$ a holomorphic map. We define its* Julia set *$\mathcal{J}$ as the closure of the repelling periodic points of f, i.e.,*

$$\mathcal{J} = \overline{\bigcup_{n \geq 1} \{z \in \mathbb{C} : f^n z = z \text{ and } |(f^n)'(z)| > 1\}}.$$

The Julia set of f is a closed invariant set.

Lemma 2.1 (Ruelle, Bowen)**.** *If $f : \mathcal{J} \to \mathcal{J}$ is expanding, there exists a Markov partition for the system.*

An important case of Julia set is the family generated by the quadratic polynomials. For $f_c(z) = z^2 + c$, call $\mathcal{J}_c$ the corresponding Julia set. A characterization of hyperbolicity of the maps f_c is given in terms of the Mandelbrot set

Definition. *We define the* Mandelbrot set *as*

$$\mathcal{M} = \{c \in \mathbb{C} : |f_c^n(0)| \not\to \infty \text{ } as n \to \infty\}.$$

Proposition 2.2. *Let f_c be a quadratic map with corresponding Julia $\mathcal{J}_c$. Then $f : \mathcal{J} \to \mathcal{J}$ is hyperbolic if and only if either c lies outside $\mathcal{M}$, or f_c has an attracting periodic point z, ie, $f_c^n z = z$ for some $n \in \mathbb{N}$ and $|(f_c^n)'(z)| < 1$. If $c \notin \mathcal{M}$, then $\mathcal{J}_\rfloor$ is a Cantor set, while if $c \in \mathcal{M}$ then $\mathcal{J}_c$ is connected.*

3 Transfer operator

We begin by introducing the notion of iterated system scheme.

Definition. *Let $U_1, \dots U_k \subset \mathbb{R}^d$ a finite collection of subsets such that $U_i = \overline{\mathrm{int}(U_i)}$. For $k \geq 2$, let A be a $k \times k$ aperiodic matrix (i.e., $A^N > 0$ for some N) with entries in $\{0,1\}$, and assume that for every pair $i, j \in \{1, \dots k\}$ of symbols such that $A_{ij} = 1$ there is an analytic map $\phi_{ji} : U_i \to U_j$ such that*

1. $\overline{\phi_{ji}(U_i)} \subset U_j$,
2. ϕ_{ji} *is a strict contraction, i.e., there exists a constant* $0 < \theta < 1$ *such that* $|(D\phi_{ji})(z)| \geq \theta$ *for every* $z \in U_i$.

   The collection $\{\phi_{ji} : A_{ij} = 1\}$ *is called an* Iterated function system.

For a string $\underline{i} = (i_1, \dots, i_{n+1}) \in \{1, \dots, k\}^n$ (for which we write $|\underline{i}| = n+1$) such that $A_{i_j, i_{j+1}} = 1$ for every $j \in \{1, \dots, n\}$, we can associate a map $\phi_{\underline{i}} = \phi_{i_{n+1} i_n} \circ \dots \circ \phi_{i_2 i_1} : U_{i_{n+1}} \to U_{i_1}$. The *Limit set* associated to an IFS is the set

$$\Lambda = \bigcap_{n=1}^{\infty} \overline{\bigcup_{|\underline{i}|=n+1} \phi_{\underline{i}}(U_{i_1})}.$$

For $\underline{i} = (i_1, \dots, i_{n+1})$ if $i_1 = i_{1+n}$ (in which case we write $\underline{i} \in \mathrm{Fix}_n$), then the contraction $\phi_{\underline{i}} : U_{i_1} \to U_{i_1}$ has a unique fixed point which we call $z_{\underline{i}}$. In this setting, the pressure function of the Bowen's equation takes the form

$$P(s) = \lim_n \frac{1}{n} \log \sum_{\underline{i} \in \mathrm{Fix}_n} |D\phi_{\underline{i}}(z_{\underline{i}})|^s.$$

An important case arise when considering Markov maps: poner como se consigue un IFS a partir de un markov map. poner como ejemplo el doubling map (mayer)

We have already seen that the Hausdorff dimension of the repeller associated to iterated function systems coming from Markov maps is given by the Bowen formula. This result also holds for general iterated function systems.

Now we introduce some operator theoretic notions etc

For every symbol $i \in \{1, \dots, k\}$ choose a polydisk $D_i = D_i^{(1)} \times \dots \times D_i^{(d)} \subset \mathbb{C}^d \geq$ such that $U_i \times \{0\} \subset D_i$. We may assume that for each admissible pair (i, j), the maps ϕ_{ji} and $|D\phi_(ji)(\cdot)|$ can be extended to maps $D_i \to D_j$ (we use the same notation for such extensions) such that

1. $\overline{\phi_{ji}(D_i)} \subset D_j$,
2. $\sup_{z \in D_i} |D\phi_{ji}(z)| < 1$.

Call $D = \coprod_{i=1}^k D_i$. For every $\underline{i} \in \mathrm{Fix_n}$, that is $i_j = i_{j+n}$ for every j, the contraction $\phi_{\underline{i}} : D_{i_1} \to D_{i_1}$ has a unique fixed point, let say $z_{\underline{i}}$ and it lies in the real section $\mathbb{R}^d$.

Now we are ready to define the transfer operator. For any open set U, let $\mathcal{A}_\infty(U)$ the Banach space of holomorphic functions on U which are bounded in $\overline{U}$, equipped with the supremum norm. For any $s \in \mathbb{C}$ and (i, j) such that $A_{ij} = 1$, define the *weight function* $w_{s,ji} \in \mathcal{A}_\infty(D_i)$ by

$$w_{s,ji}(z) = |D\phi_{ji}(z)|^s.$$

Then define the bounded linear operator $\mathcal{L}_{s,ji} : \mathcal{A}_\infty(D_j) \to \mathcal{A}_\infty(D_i)$ by

$$\mathcal{L}_{s,ji} g(z) = g(\phi_{ji} z) w_{s,ji}(z).$$

Summing over all the symbols j compatible with i, we have the *component transfer operator*,

$$\mathcal{L}_{s,i} h(z) = \sum_{j:A_{ij}=1} \mathcal{L}_{s,ji} h(\phi_{ji}(z)) = \sum_{j:A_{ij}=1} h(\phi_{ji} z) w_{s,ji}(z).$$

The operator extends naturally to an operator $\mathcal{A}_\infty(\coprod_{j:A_{ij}=1} D_j) \to \mathcal{A}_\infty(D_i)$ and more over, it extends to an operator $\mathcal{A}_\infty(D) \to \mathcal{A}_\infty(D_i)$. Finally, we fine the transfer operator $\mathcal{L}_s : A_\infty(D) \to A_\infty(D)$ by

$$\mathcal{L}_s h|_{D_i} = \mathcal{L}_{s,i} h$$

for each $h \in \mathcal{A}_\infty(D)$ and $i \in \{1, \ldots, k\}$.

example: doubling map, gauss map

The following result is one of the key aspects connecting the transfer operator to thermodynamic formalism

Theorem 3.1 (Ruelle)**.** *For real s, the transfer operator $\mathcal{L}_s; \mathcal{A}_\infty(D) \to \mathcal{A}_\infty(D)$ has spectral radius* $\exp(P(s))$*, being this the unique eigenvalue of maximum modulus, and it is simple and isolated.*

This result, together with perturbative methods of operators, allows us to obtain information about the regularity of the function coding the Hausdorff dimension of certain limit sets.

examples?

More information of the spectrum of $\mathcal{L}$ is known when acting on the space $\mathcal{A}_\infty(D)$. We introduce the notion of nuclear operators, due to Grothendieck

Definition. *A linear operator $L : B \to B$ on a Banach space is said to be* nuclear of order p *if there exist $\{u_n\} \subset B$, $\{l_n\} \subset B^*$ (with $\|u_n\| = \|l_n\| = 1$ and $\{\rho_n\} \subset \mathbb{C}$ with $\sum_n |\rho_n|^p < \infty$ such that*

$$L(v) = \sum_{n=0}^{\infty} \rho_n l_n(v) u_n$$

for all $v \in B$. If L is nuclear of order p for every $p > 0$, then we say that L is nuclear of order zero.

Note that nuclear operators are compact. One of the the properties of nuclear operators is that their trace

$$\mathrm{Tr}(L) = \sum_{\lambda \text{ eigenvalues}} \lambda$$

is well defined.

example: restriction operator

The following result allows us to fully exploit the properties of the spectrum of the transfer operator:

Theorem 3.2 (Ruelle). *The transfer operator $\mathcal{L} : \mathcal{A}_\infty(D) \to \mathcal{A}_\infty(D)$ is nuclear of order zero.*

The trace of the transfer operator and its iterates can be explicitly computed;

Proposition 3.3. *If $\mathcal{L}_s : \mathcal{A}_\infty(D) \to \mathcal{A}_\infty(D)$ is the transfer operator associated to a conformal IFS scheme then*

$$\mathrm{Tr}(\mathcal{L}_s^n) = \sum_{\underline{i} \in \mathrm{Fix}_n} \frac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - D\phi_{\underline{i}}(z_{\underline{i}}))}.$$

With the above proposition we can write the Fredholm determinant of the transfer operator as

$$\begin{aligned} \det(I - z\mathcal{L}_s) := & \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Tr}\, \mathcal{L}_s^n\right) \\ = & \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{\underline{i} \in \mathrm{Fix}_n} \frac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - D\phi_{\underline{i}}(z_{\underline{i}}))}\right) \end{aligned}$$

Once more, we go back to the works of Grothendieck

Proposition 3.4 (Grothendieck). *The Fredholm determinant $\det(I - z\mathcal{L}_s)$ is an entire function of both s and z. If $\lambda_r(s), r = 1, 2, \dots$ is an enumeration of the eigenvalues of $\mathcal{L}_s$, counted with multiplicity, then*

$$\det(I - z\mathcal{L}_s) = \prod_{r=1}^{\infty} (1 - z\lambda_r(s)).$$

As a consequence of all the above, we obtain

Theorem 3.5. *The Hausdorff dimension of the limit set Λ associated to an IFS is the largest zero of the function $z \mapsto \det(I - z\mathcal{L}_s)$.*

Since the Fredholm determinant is an entire function, it admits a power series expansion

$$\det(I - z\mathcal{L}_s) = 1 + \sum_{N=1} d_N(s) z^N$$

Comparing with the product formula for the Fredholm determinant, it is possible to determine explicitly the coefficients of such expansion.

Proposition 3.6. *The coefficients $d_N(s)$ are given by*

$$d_N(s) = \sum_{\substack{(n_1, \dots, n_m) \\ n_1 + \dots n_m = N}} \frac{(-1)^m}{m!} \prod_{l=1}^{m} \frac{1}{n_l} \sum_{\underline{i} \in \mathrm{Fix}_{n_l}} \frac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{\det(I - D\phi_{\underline{i}}(z_{\underline{i}}))}.$$

There exists $0 < \delta < 1$ such that $d_N(s) = O(\delta^{N^{1+1/d}})$ as $N \to \infty$ for all $s > 0$.

Once the fixed points of the mappings ϕ are known, the above expression allows us to efficiently evaluate the coefficients of the power series expression for the Fredholm determinant.

References