On the central limit theorem for stationary processes

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This note shows that the central limit theorem (CLT) for some classes of stationary (in the narrow[1] sense of random processes), can be obtained from CLT for stationary martingale-difference sequences.

Suppose that a space X is given with an σ-algebra of sets M and a probability measure P. The spaces L_p correspond to the measure P; $|f|_p$ is the norm of the function f in L_p.

If a σ-algebra L is contained in M, then $H(L)$ denotes the Hilbert space of those functions from L_2 that are measurable with respect to L.

The symbol P_G denotes an orthogonal projector on a closed subspace $G \subset H = L_2$.

Let T be an ergodic automorphism of a space X with measure P, M_0 a σ-algebra such that $T^{-1}(M_0) \subset M_0$. The relation $Uf(x) = f(Tx)$ defined in H is a unitary operator.

Finally, set $M_k = T^{-k}(M_0)$, $H_k = H(M_k) = U^k(H_0)$, $S_k = H_k \ominus H_{k+1}$[2]. Let R denote the linear subspace of elements $g \in H$ with $g \in H_k \ominus H_\ell$ for some k and ℓ with $-\infty < k \leq \ell < \infty$.

Theorem 1. *Let $f \in L_2$ and*

$$\inf_{g \in R} \overline{\lim_{n \to \infty}} \frac{|\sum_{k=0}^{n-1} U^k(f-g)|_2}{\sqrt{n}} = 0.$$

Then there exists

$$\sigma = \lim_{n \to \infty} |\sum_{k=0}^{n-1} U^k f|_2 / \sqrt{n} \tag{1}$$

and

$$P\{\sum_{k=0}^{n-1} U^k f/\sqrt{n} < z\} \xrightarrow[n \to \infty]{} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2\sigma^2} du. \tag{2}$$

[1] Weak

[2] Here $B \ominus A = \{x \in B : (x,y) = 0 \text{ for all } y \in A\}$ for subspaces $A \subset B$.

We outline the proof of the theorem. Let $\varepsilon_p > 0$, $\varepsilon_p \xrightarrow[p\to\infty]{} 0$, $f_p \in R$ and

$$\overline{\lim_{n\to\infty}} |\sum_{k=0}^{n-1} U^k (f - f_p)|_2/\sqrt{n} < \varepsilon_p.$$

Consider the chain of equalities

$$f = f_p + f - f_p = \sum_{\ell=-\infty}^{\infty} P_{S_\ell} f_p + f - f_p = \sum_{\ell=-\infty}^{\infty} U^{-\ell} P_{S_\ell} f_p$$

$$+ \sum_{\ell=-\infty}^{\infty} \sum_{m=0}^{-\ell-1} U^m P_{S_\ell} f_p - U \sum_{\ell=0}^{\infty} \sum_{m=0}^{-\ell-1} U^m P_{S_\ell} f_p + f - f_p = h_p + g_p - U g_p + f - f_p.$$

Furthermore,

$$\overline{\lim_{n\to\infty}} |\sum_{k=0}^{n-1} U^k (f - h_p)|_2/\sqrt{n} \leq$$

$$\overline{\lim_{n\to\infty}} \{|h_p - U^n h_p|_2 + |\sum_{k=0}^{n-1} U^k (f - f_p)|_2\}/\sqrt{n} < \varepsilon_p \tag{3}$$

Notice now that $\sum_{\ell=-\infty}^{\infty} U^{-\ell} P_{S_\ell} f_p \in S_0$, $U^k f_p \in S_k$. This implies that $U^k h_p$ is measurable with respect to M_k and orthogonal $H_{k+1} = H(M_{k+1})$, i.e., that the sequence $U^{-k} h_p$ is an ergodic martingale-difference sequence. Therefore [1], [2], $\sum_{k=0}^{n-1} U^k h_p/\sqrt{n}$ are asymptotically normally distributed with variance $\sigma^2 = |h_p|_2$ [2].

The sequence σ_p converges to some limit σ since

$$|\sigma_p - \sigma_{p'}| \leq |h_p - h_{p'}|_2 \leq |\sum_{k=0}^{n-1} U^k (h_p - h_{p'})|_2/\sqrt{n}$$

$$\overline{\lim_{n\to\infty}} \{|\sum_{k=0}^{n-1} U^k (f - h_p)|_2/\sqrt{n} + |\sum_{k=0}^{n-1} U^k (f - h_{p'})|_2/\sqrt{n}\} \leq \varepsilon_p + \varepsilon_{p'}.$$

The relation (2) now follows from Lemma 5.3 of [4], equality 1 follows from (3) and the fact that $\sigma_p \xrightarrow[p\to\infty]{} \sigma$.

Theorem 2. *Let T be a ergodic automorphism, M_0 the same σ-algebra as in theorema 1; $f \in L_{2+\delta}$ for some $0 \leq \delta \leq \infty$ and*

$$\sum_{A\geq 0} (|P_{H_A} f|_{(2+\delta)/(1+\delta)} + |f - P_{H_{-A}} f|_{(2+\delta)/(1+\delta)}) < \infty.$$

Then the condition of Theorem 1 is satisfied.

Let us explain how Theorem 2 is proved. Denote $\int_X g(x)h(x)P(dx)$ instead by (g,h). Set

$$f_1^{(A)} = P_{H_A}f, \quad f_2^{(A)} = f - P_{H_{-A}}f, \quad r_i^{(A)}(k) = (f_i^{(A)}, U^k f_i^{(A)}) \qquad (i=1,2).$$

From Hölder's inequality and the well-known conditional expectations property (for instance, [3] p 508) it follows that

$$|r_i^{(A)}(k)| = |(f_i^{(A)}, U^{|k|} f_i^{(A)})| = |(f_i^{(A)}, U^{\pm|k|} f_i^{(A\pm|k|)})| \le$$
$$|f_i^{(A)}|_{2+\delta} |f_i^{(A\pm|k|)}|_{(2+\delta)/(1+\delta)} \le 2|f_i^{(A)}|_{2+\delta} |f_i^{(A\pm|k|)}|_{(2+\delta)/(1+\delta)}.$$

Using this estimate, we find that

$$\varlimsup_{n\to\infty} |\sum_{k=0}^{n-1} U^k(f_1^{(A)}) + f_2^{(A)}|_2/\sqrt{n} \le \sum_{i=1}^{2} \varlimsup_{n\to\infty} |\sum_{k=0}^{n-1} U^k f_i^{(A)}|_2/\sqrt{n} =$$

$$\sum_{i=1}^{2} \lim_{n\to\infty} \left(\sum_{|k|<n} \left(1 - \frac{|k|}{n}\right) r_i^{(A)}(k) \right)^{1/2} \le 2|f|_{2+\delta}^{1/2} \sum_{i=1}^{2} \left(\sum_{k=A}^{\infty} |f_i^{(k)}|_{(2+\delta)/(1+\delta)} \right)^{1/2} \xrightarrow[A\to\infty]{} 0.$$

It remains to be noted that

$$f - f_1^{(A)} - f_2^{(A)} \in R.$$

Remark 1. *Theorems 1 and 2 can be reformulated for the case when T is an endomorphism. In this case, the spaces S_k form a one-way infinite sequence.*

Remark 2. *Under the conditions of Theorem 2, the equality $f = g + Uh - h$, where $g, h \in L_{(2+\delta)/(1+\delta)}$, g measurable with respect to M_0 and has zero integral over all sets in M_1. Such representation is useful in the proof of finer limit theorems, for example, in proving the weak convergence of distributions in the space of continuous functions corresponding to the sequences of random broken lines constructed from the sums $\sum_{k=0}^{n-1} U^k f$, to the process corresponding to the Brownian motion.*

Remark 3. *From Theorem 2 it is easy to obtain a series of theorems concerning processes with strong and uniformly strong mixing and functionals from such processes.*

In particular, Theorems 18.6.1, 18.6.2 and 18.6.3 from the book [3] are consequences of Theorem 2. In addition, under the conditions of Theorem 18.6.1 and some strengthening of the conditions of the theorems 18.6.2 and 18.6.3 of [3] yield the representation $f = g + Uh - h$, $g \in S_0$, $h \in L_2$.

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