Clase 8:

Integrales definidas y sustitución:

Ej: $\int_0^1 e^{5x}\,dx$

1. Sacando la anti-derivada del integrando

$u = 5x$

$\frac{du}{5} = dx$

$$\int e^{5x}\,dx = \int e^u \frac{du}{5} = \frac{1}{5}e^u + C$$

$$= \frac{1}{5}e^{5x} + C$$

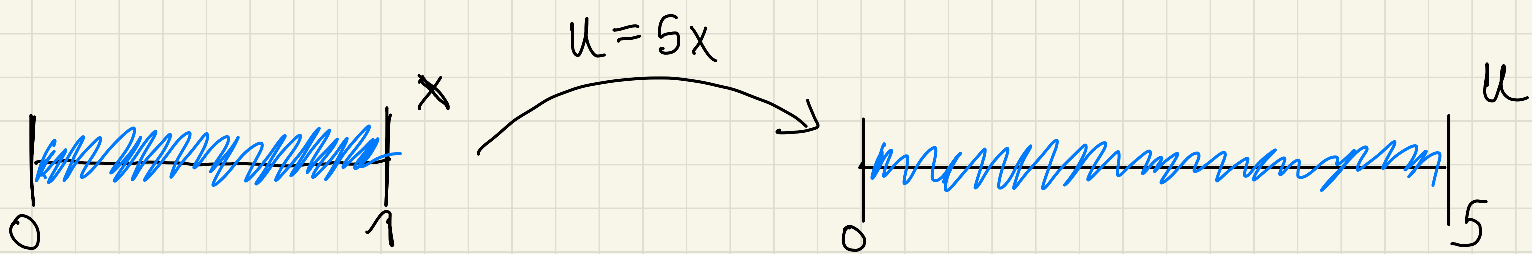
Luego, uno evalua esto en los límites de integración:

$$\int_0^1 e^{5x}\,dx = \frac{1}{5}e^{5x}\Big|_0^1 = \frac{1}{5}e^5 - \frac{1}{5}$$

2. Otra forma: cambiar el intervalo de integración al hacer la sustitución:

$$\int e^{5x}\,dx \xrightarrow{u=5x} \int \frac{e^u}{5}\,du$$

aplico eso al intervalo de integración



$$\int_0^1 e^{5x}\,dx = \frac{1}{5}\int_0^5 e^u\,du = \frac{1}{5}\left(e^u\right)\Big|_0^5$$

$$= \frac{1}{5}e^5 - \frac{1}{5}$$

Ej: $\int_1^2 3x\sqrt{2x^2+7}\,dx$

" u = lo más raro/complicado "

$u = 2x^2 + 7$

$3\frac{du}{4} = 3x\,dx$

x	$u = 2x^2+7$
1	9
2	15

← nuevos límites de integración

$$= \int_9^{15} \sqrt{u}\,\frac{3}{4}\,du$$

$$= \frac{3}{4}\int_9^{15} u^{1/2}\,du = \frac{\cancel{3}}{\cancel{4}\,2}\,\frac{u^{3/2}}{\cancel{3}/\cancel{2}}\Bigg|_9^{15} = \boxed{\frac{1}{2}\left(15^{3/2}-9^{3/2}\right)}$$

Ej: $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

$u = x$
$du = dx$ $\Rightarrow$ no conduce a nada

$u = \ln x$
$du = 1/x\,dx$

$x = 1 \Rightarrow u = \ln 1 = 0$
$x = e \Rightarrow u = \ln e = 1$

$$= \int_0^1 u\,du = \frac{u^2}{2}\Bigg|_0^1 = \frac{1}{2}$$

Teorema:
Si g' es continua en $[a,b]$ y f es continua en

al rango de $u(x) = g(x)$, entonces

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Este teorema sólo dice lo que hemos estado haciendo; hay que saber usarlo.

Simetría:

Asuma que f es continua en $[-a, a]$.

Decimos que f es par si

$$f(-x) = f(x) \quad , \quad x \in [-a,a]$$

f es impar si

$$f(-x) = -f(x) \quad , \quad x \in [-a,a]$$

Ejemplos:

- $f(x) = x^2 \quad , \quad [-1,1]$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

f es par

- $f(x) = x^3$, $[-1,1]$

$$f(-x) = (-x)^3 = (-x)(-x)^2 = (-x)x^2$$
$$= -x^3 = -f(x)$$

f es impar.

$f(x) = x^a$, a natural es par si

a es par, impar si a es impar.

Funciones impares:

- $\operatorname{sen} x$
- $\tan x$
- $1/x$ (delicado, porque se indefine en 0, $[-a, a] \setminus \{0\}$)

$$f(x) = 1/x \Rightarrow f(-x) = -f(x), \ x \neq 0.$$

- $\operatorname{arcsen} x$

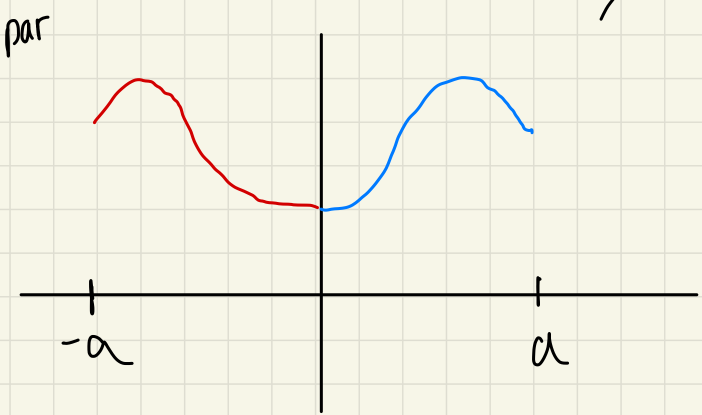
⋮

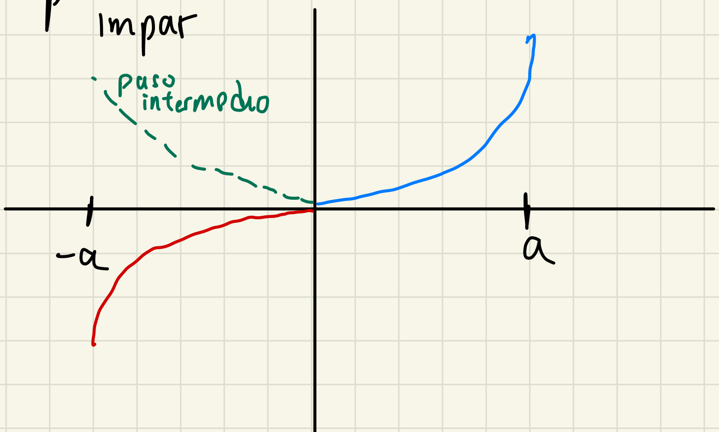
Funciones pares

- $\cos x$
- x^2
- $|x|$

$\vdots$

Graficamente, esto representa:





Para las integrales, esto significa:

- f par, $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

- f impar, $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$

Ej: $\displaystyle\int_{-1}^{1} x^6+1\,dx$

$f(x) = x^6+1$

$f(-x) = (-x)^6+1 = x^6+1 = f(x)$ par

$$\int_{-1}^{1} x^6+1\,dx = 2\int_{0}^{1} x^6+1\,dx = 2\left(\frac{x^7}{7}+x\right)\Bigg|_0^1$$

$$= 2\left(\frac{1}{7}+1\right)$$

$$\int_{-6}^{6} \frac{\tan x}{1+x^2+x^4}\,dx$$

$$f(x) = \frac{\tan x}{1+x^2+x^4}, \quad f(-x) = \frac{\tan(-x)}{1+x^2+x^4}$$

$$= \frac{-\tan(x)}{1+x^2+x^4}$$

$$= -f(x)$$

f es impar

$$\int_{-6}^{6} \frac{\tan x}{1+x^2+x^4}\,dx = 0$$

Ejercicios:

1. Calcule $\displaystyle\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \sqrt{u}\,\frac{du}{2}$

Solución: $u = 2x+1$ nuevos límites??

$du = 2dx$ $u = 1, 9$

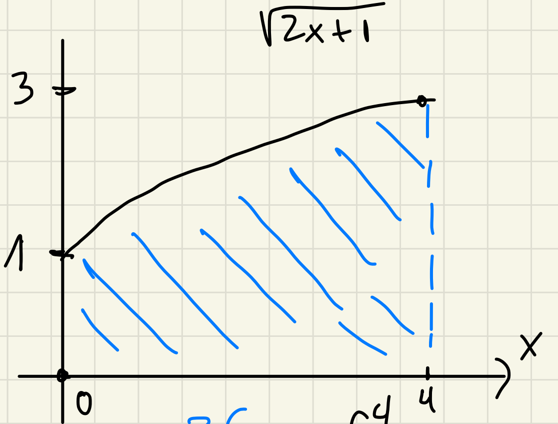
$$\frac{1}{2}\int_1^9 u^{1/2}\,du = \frac{1}{2}\frac{u^{3/2}}{3/2}\Big|_1^9 = \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right)$$

$$= \frac{1}{3}(27-1) = \frac{26}{3}$$

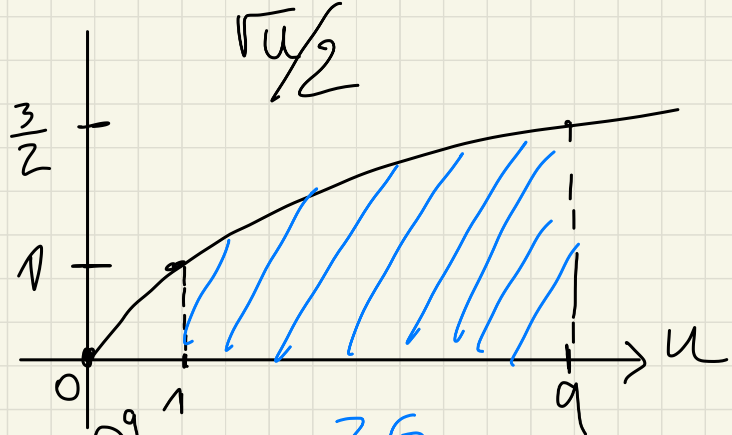
$x^{1/2} = \sqrt{x}$

$x^{3/2} = (\sqrt{x})^3$

Obs: $9^{3/2} = (\sqrt{9})^3 = 3^3 = 27$



$\frac{26}{3} = \int_0^4 \sqrt{2x+1}\, dx$



$= \frac{1}{2}\int_1^9 \sqrt{u}\, du \quad \frac{26}{3}$

Ej: $\int_1^2 \frac{e^{1/x}}{x^2}\, dx$

$u = 1/x = x^{-1}$

$-du = 1/x^2 \, dx$

límites de la integral: $\begin{matrix} x = 1 \\ u = 1 \end{matrix} \qquad \begin{matrix} x = 2 \\ u = 1/2 \end{matrix}$

$$\int_a^b = -\int_b^a$$

$$= -\int_1^{1/2} e^u \, du = \int_{1/2}^{1} e^u \, du = e^u \Big|_{1/2}^{1} = e - e^{1/2}$$

Ej: $\displaystyle\int \frac{x}{1+x^4}\,dx$

$u = x^2 \Rightarrow u^2 = x^4$

$du = 2x\,dx \qquad 1+u^2 = 1+x^4$

$$= \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan(u) + C$$

$$= \frac{1}{2}\arctan(x^2) + C$$

$$\int \frac{1}{1+x^2}\,dx = \text{Regalo de puntos}$$

$$\int_{-\pi/2}^{\pi/2} \text{sen}^3(x)\,dx = 0$$

$$f(x) = \text{sen}^3(x) \qquad f(-x) = (\text{sen}(-x))^3 = (-\text{sen}\,x)^3$$

$$= -\text{sen}^3(x)$$

$$= -f(x)$$

Olvidamos que sabemos el argumento de simetría:

$$\int \operatorname{sen}^3(x)\,dx = \int \operatorname{sen}^2(x)\cdot \operatorname{sen}(x)\,dx$$

$$\operatorname{sen}^2(x) + \cos^2(x) = 1$$

$$\operatorname{sen}^2(x) = 1 - \cos^2(x)$$

$$\int (1-\cos^2 x)\operatorname{sen}(x)\,dx$$

$$= \int \operatorname{sen}(x)\,dx - \int \cos^2(x)\operatorname{sen}(x)\,dx$$

$u = \cos(x)$ $du = -\operatorname{sen}(x)dx$

$$= -\cos(x) + C + \int u^2\,du$$

$$= -\cos(x) + C + \frac{u^3}{3}$$

$$= -\cos(x) + C + \frac{\cos^3(x)}{3}$$

$$\int_{-\pi/2}^{\pi/2} \operatorname{sen}^3(x)\,dx = \left(-\cos(x) + \frac{\cos^3(x)}{3}\right)\Bigg|_{-\pi/2}^{\pi/2}$$

$$= -\cos(\pi/2) + \frac{\cos^3(\pi/2)}{3} - \left(-\cos(-\pi/2) + \frac{\cos^3\left(\frac{-\pi}{2}\right)}{3}\right)$$

$$= -\cos(\pi/2) + \frac{\cos^3(\pi/2)}{3} - \left(-\cos(\pi/2) + \frac{\cos^3}{3}\left(\frac{\pi}{2}\right)\right)$$

algo $-$ (algo) $= 0$

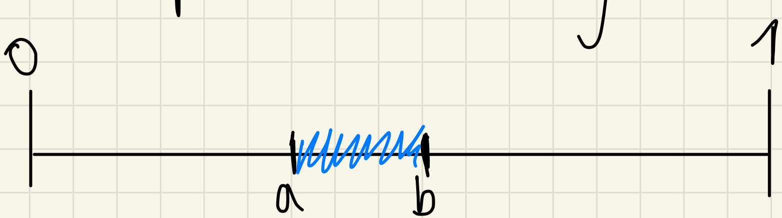
$$\cos(-x) = \cos(x)$$

Problema:

Suponga que tiene una barra de largo 1m. La densidad de la barra está dada por

$$f(x) = \frac{dm}{dx} = 3x^2 \quad , \quad x \in [0,1]$$

1. Encuentre la masa total de la barra.
2. Encuentre el punto de la barra que la divide en dos partes de igual masa.



masa entre a y b es $= \displaystyle\int_a^b \frac{dm}{dx}\,dx$