Clase 7: sustitución

Recordar el TFC: Si $F' = f$

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a)$$

Dada una f, pensar en F a veces es difícil. Necesitamos técnicas para eso.

Ejemplo: $\int e^{x^2} x\,dx = \frac{1}{2} e^{x^2} + C$

cómo se me ocurre eso??

$$\int_a^b x e^{x^2}\,dx = \int_a^b \frac{1}{2} \cdot 2x\, e^{x^2}\,dx = \int_a^b \frac{1}{2} \left(e^{x^2}\right)' dx$$

$$= \frac{1}{2} e^{x^2} \Big|_a^b = \frac{1}{2} e^{b^2} - \frac{1}{2} e^{a^2}$$

Recordemos la regla de la cadena:

$$\frac{d}{dx} (f (g(x))) = f'(g(x)) \cdot g'(x)$$

$$\int_a^b f'(g(x)) \cdot g'(x) dx = \int_a^b \frac{d}{dx} (f(g(x))) dx = \int_a^b (f(g(x)))' dx$$

$$= f(g(x)) \Big|_a^b = f(g(b)) - f(g(a))$$

Como integral indefinida

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + C$$

$$u = g(x)$$

$$\frac{du}{dx} = g'(x) \rightsquigarrow du = g'(x)\,dX$$

$$\int f'(u)\,du = f(u) + C = f(g(x)) + C$$

Ejemplo:

$$\int x e^{x^2}\,dx$$

$$u = x^2$$

$$\frac{du}{dx} = 2x$$

$$\frac{du}{2} = x\,dx$$

$$= \int e^{\overset{=u}{x^2}} \underbrace{x\,dx}_{=\frac{du}{2}} = \int e^{u} \frac{du}{2} = \frac{1}{2}\int e^{u}\,du$$

$$= \frac{1}{2} e^{u} + C \overset{u = x^2}{=} \frac{1}{2} e^{x^2} + C$$

Ej: $\int x^3 \cos(\underbrace{x^4+2}_{=u})\,dx = I$

$u = x^4 + 2$

$\frac{du}{dx} = 4x^3$

$\frac{du}{4} = x^3\,dx$

$$I = \int \cos(\underbrace{x^4+2}_{u})\,\underbrace{x^3\,dx}_{du/4}$$

Sugerencia:

" u = lo que se vea más complicado "

ojo: no siempre funciona

$\int \cos(x)\,dx = \sin x + c$

$\int \cos(\text{algo raro})\,dx = ??$

$$= \frac{1}{4}\int \cos(u)\,du = \frac{1}{4}\operatorname{sen}(u) + C$$

$$= \frac{1}{4}\operatorname{sen}(x^4+2) + C$$

Ej: $\int e^{5x}\,dx$

$5x = u$

$\frac{du}{dx} = 5$

$\frac{du}{5} = dx$

$\int e^x\,dx = e^x + C$

$$= \int e^u \frac{du}{5} = \frac{1}{5}\int e^u\,du = \frac{1}{5}e^u + C$$

$$= \frac{1}{5}e^{5x} + C$$

$$\int e^{x} dx = e^{x} + c$$

$$e^{x+1} = e^{x} \cdot e^{1}$$

Ej: $\int e^{x+1} dx = \int e^{x} \cdot e \, dx$

$$= e \int e^{x} dx = e(e^{x} + c)$$

$$= e^{x+1} + \tilde{C}$$

ambos métos sirven y son válidos

$u = x+1$
$du = dX$

$$\int e^{x+1} dx = \int e^{u} du = e^{u} + c$$

$$= e^{x+1} + c$$

Ej: $\int \sqrt{1+x^2}\, x^5 \, dx = I$

opciones
1. $u = 1+x^2$
2. $u = x^5$

Sustitución 1: $u = 1+x^2$

$\frac{du}{dx} = 2x \qquad du = \underline{2x\,dx}$

$x^5 = x^4 \cdot x$

$$I = \int \underbrace{\sqrt{1+x^2}}_{=u}\, \underbrace{x^5 dx}_{= ??} = \int \sqrt{\underbrace{1+x^2}_{=u}}\, \underbrace{x^4}_{=(u-1)^2} \cdot \underbrace{x\,dx}_{\frac{du}{2}}$$

no puedo

$\cancel{\int \sqrt{u}\, x^4 \frac{du}{2}}$

Queremos escribir x^4 en términos de u

$$u = 1+x^2 \Rightarrow u-1 = x^2 \Rightarrow (u-1)^2 = x^4$$

$$I = \int \sqrt{u} \cdot (u-1)^2 \cdot \frac{du}{2} = \frac{1}{2}\int u^{1/2}(u^2-2u+1)\,du$$

$$\int u^{\text{algo}}\,du + \int u^{\overset{\text{otro}}{\text{algo}}}\,du$$

$$= \frac{1}{2}\int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right)du$$

$$= \frac{1}{2}\left(\int u^{5/2}\,dx - 2\int u^{3/2}\,du + \int u^{1/2}\,du\right)$$

$$= \frac{1}{2}\left(u^{7/2}/_{7/2} - 2\,u^{5/2}/_{5/2} + u^{3/2}/_{3/2}\right) + C$$

$$= \frac{1}{7}u^{7/2} - \frac{2}{5}u^{5/2} + \frac{1}{3}u^{3/2} + C$$

$$u = x^2+1$$

$$= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C$$

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Ej: $\int (x+3)(x-1)^{1/2}\,dx = \int \underbrace{(x+3)}_{u+4}\underbrace{\sqrt{x-1}}_{u}\,\underbrace{dx}_{du}$

$u = x-1 \rightsquigarrow x = u+1$

$\frac{du}{dx} = 1 \qquad x+3 = u+1+3 = u+4$

$du = dx$

$$= \int (u+4)\sqrt{u}\,du = \int (u+4)u^{1/2}\,du = \int u^{3/2} + 4u^{1/2}\,du$$

$$= \int u^{3/2}\,du + 4\int u^{1/2}\,du$$

$$= u^{5/2}/5/2 + 4\,u^{3/2}/3/2 + C$$

$$= \frac{2}{5}u^{5/2} + \frac{8}{3}u^{3/2} + C \qquad u = x-1$$

$$= \frac{2}{5}(x-1)^{5/2} + \frac{8}{3}(x-1)^{3/2} + C$$

Ej: $\int \tan x \, dx$ $\tan x = \dfrac{\operatorname{sen} x}{\cos x}$

$= \int \dfrac{\operatorname{sen} x}{\cos x} dx$

candidatos para la u:

1. $\operatorname{sen} x$
2. $\cos x$

Sustitución "incorrecta":

$u = \operatorname{sen} x$

$\dfrac{du}{dx} = \cos x \qquad du = \cos x \, dx$

no es lo que queremos

$\int \dfrac{\operatorname{sen} x}{\cos x} dx$ nos queda $\operatorname{sen} x \cdot \dfrac{dx}{\cos x}$

Sustitución "correcta":

$$\int \frac{1}{u}\,du = \ln|u|$$

valor absoluto

$$u = \cos x$$

$$\frac{du}{dx} = -\operatorname{sen} x \qquad -du = \operatorname{sen} x\,dx$$

$$\int \frac{\operatorname{sen} x\,dx}{\cos x} = \int -\frac{du}{u} = -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C = \underline{-\ln|\cos x|} + C$$

$$= \ln|(\cos x)^{-1}| + C = \underline{\ln|\sec x|} + C$$

ambos están bien

Ej: $\displaystyle\int \frac{1+e^x}{1-e^x}\,dx$

$$\frac{1+e^x}{1-e^x} = \frac{1-e^x+2e^x}{1-e^x}$$

$$= \frac{1-e^x}{1-e^x} + \frac{2e^x}{1-e^x}$$

$$= 1 + \frac{2e^x}{1-e^x}$$

$$\int \frac{1+e^x}{1-e^x}\,dx = \int 1\,dx + \int \frac{2e^x}{1-e^x}\,dx$$

Hicimos la integral:

$$\int \frac{1-x^2}{1+x^2}\,dx$$

Idea:

$$\frac{1-x^2}{1+x^2} = \frac{1}{1+x^2} - 2$$

$$= x + C + 2\underbrace{\int \frac{e^x}{1-e^x}\,dx}$$

$$\int \frac{e^x}{1-e^x}\,dx$$

" u = lo más raro / complicado "

$$u = 1 - e^x$$
$$-du = e^x\,dx$$

$$= \int \frac{-du}{u} = -\int \frac{1}{u}\,du$$
$$= -\ln|u| + C$$
$$= -\ln|1-e^x| + C$$

$$\int \frac{1+e^x}{1-e^x}\,dx = x + C + 2\left(-\ln|1-e^x| + C\right)$$

$$= x - 2\ln|1-e^x| + C$$

Recomendación: intentar

$$\int \frac{1+e^x}{1-e^x}\,dx = \int \frac{1}{1-e^x}\,dx + \int \frac{e^x}{1-e^x}\,dx$$

Qué tan difícil es esta integral??

Spoilers: es más difícil, pero sí se puede hacer. Intentar ver por qué.

$1-e^x = u$