Clase 17: fracciones parciales

Idea

$$\int \frac{p(x)}{q(x)}\,dx \qquad p \text{ y } q \text{ polinomios.}$$

· Casos que vimos eran tales que

- $\text{Grado}(p) < \text{Grado}(q)$
- q no tenía raíces repetidas

Hoy vamos a ver qué pasa cuando esto no

Se cumple.

Ej: $\int \frac{dx}{x^2-a^2}$ $\quad a \neq 0$

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$

$$= \frac{A(x+a) + B(x-a)}{(x-a)(x+a)}$$

$$\frac{1}{x^2-a^2} = \frac{x(A+B) + a(A-B)}{(x-a)(x+a)}$$

$$A + B = 0 \qquad A = -B \qquad A = \frac{1}{2a}$$

$$a(A - B) = 1 \qquad a \cdot 2A = 1 \qquad B = -\frac{1}{2a}$$

De aquí, nuestra fracción original es

$$\frac{1}{x^2 - a^2} = \frac{1}{2a} \cdot \frac{1}{x-a} - \frac{1}{2a} \cdot \frac{1}{x+a}$$

Ahora integramos ambos lados:

$$\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \int \frac{dx}{x-a} - \frac{1}{2a} \int \frac{dx}{x+a}$$

$$= \frac{1}{2a} \ln|x-a| - \frac{1}{2a} \ln|x+a| + C$$

$$= \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

Pregunta: que pasa si el grado de arriba es mayor al de abajo??

Ej: $\int \frac{x^3+x}{x-1} dx$

Podemos observar que

$$x^3 + x = (x-1)\left(x^2 + x + 2 + \frac{2}{x-1}\right)$$

Pregunta del millón: de dónde sale esto? cómo se me ocurre?

Dos respuestas:

1. División de polinomios

$$(x^3 + x) : (x - 1) = x^2 + x + 2$$

$$-(x^3 - x^2)$$

$$\cancel{} \quad x^2 + x$$

$$-(x^2 - x)$$

$$\cancel{} \quad 2x$$

$$-(2x - 2)$$

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Esto nos dice que

$$(x^3+x) = \left(x^2+x+2+\frac{2}{x-1}\right)(x-1)$$

2. Usar una calculadora:

Wolframalpha: $(x^3+x)/(x-1)$

$$\rightsquigarrow \left(x^2+x+2+\frac{2}{x-1}\right)$$

Luego

$$\int \frac{x^3+x}{x-1}\,dx = \int x^2+x+2+\frac{2}{x-1}\,dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$$

Pregunta: caso raíces repetidas en el denom.

Ej: $\displaystyle\int \frac{4x}{x^3 - x^2 - x + 1}\,dx$

Factorizamos el denominador

Wolfram alpha: factor $(x^3 - x^2 - x + 1)$

$$x^3 - x^2 - x + 1 = (x-1)^2(x+1)$$

Otra opción: hacerlo a mano

nunca había pasado eso

Obs: denom $= (x-1)^2(x+1)$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$\frac{4x}{\%} = \frac{A(x-1)(x+1) + B(x+1) + C(x-1)^2}{(x-1)^2(x+1)}$$

$$= \frac{x^2(A+C) + x(B-2C) + (-A+B+C)}{}$$

$$\boxed{\begin{aligned} A+C &= 0 \\ B-2C &= 4 \\ -A+B+C &= 0 \end{aligned}}$$

$$A = 1$$
$$B = 2$$
$$C = -1$$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}$$

$$\int \frac{4x}{(x-1)^2(x+1)}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{2\,dx}{(x-1)^2} - \int \frac{1}{1+x}\,dx$$

$$= \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$

$$2\int \frac{dx}{(x-1)^2} \qquad u = x-1 \qquad du = dx \qquad 2\int \frac{du}{u^2} = -2\,\frac{1}{u} = \frac{-2}{x-1}$$

Ej: $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$

Factorizamos el denom:

$$x^3 + 4x = x(x^2+4)$$

no se puede factorizar más

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

De aquí en adelante, todo es igual

$$= \frac{A(x^2+4) + (Bx+c)x}{x(x^2+4)}$$

$$\frac{2x^2 - x + 4}{\%} = \frac{x^2(A+B) + xC + 4A}{\%}$$

$A + B = 2 \qquad C = -1 \qquad 4A = 4$

$A = 1$, $B = 1$, $C = -1$

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{1}{x} + \frac{x-1}{x^2+4}$$

Ahora integramos:

$$\int LD\, dx = \int \frac{1}{x}\, dx + \int \frac{x-1}{x^2+4}\, dx$$

$$= \ln|x| + \int \frac{x-1}{x^2+4}\, dx$$

$$\int \frac{x-1}{x^2+4}\,dx = \underbrace{\int \frac{x}{x^2+4}\,dx}_{I_1} - \underbrace{\int \frac{dx}{x^2+4}}_{I_2}$$

$u = x^2+4$
$du = 2x\,dx$

$$I_1 = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+4|$$

$$= \frac{1}{2}\ln(x^2+4)$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x$$

$$I_2 = \int \frac{dx}{x^2+4} = \frac{1}{4}\int \frac{dx}{\frac{x^2}{4}+1}$$

$$= \frac{1}{4}\int \frac{dx}{\left(\frac{x}{2}\right)^2+1} \qquad u = \frac{x}{2} \qquad 2du = dx$$

$$du = \frac{dx}{2}$$

$$= \frac{1}{4}\int \frac{2\,du}{u^2+1} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\arctan(u)$$

$$= \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

Nuestra integral original

$$= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

Ej: $\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx$

$$\frac{4x^2-3x+2}{4x^2-4x+3} = 1 + \frac{x-1}{4x^2-4x+3}$$

$$\int \%\,dx = \int dx + \underbrace{\int \frac{x-1}{4x^2-4x+3}\,dx}_{I_1}$$

Cómo calculamos I_1 ?!

$$I_1 = \int \frac{x-1}{4x^2-4x+3}\,dx = \underbrace{\int \frac{x}{4x^2-4x+3}}_{I_2} - \underbrace{\int \frac{1\,dx}{4x^2-4x+3}}_{I_3}$$

$$I_2 = \int \frac{x}{4x^2-4x+3}\,dx$$

$$\cancel{u = 4x^2-4x+3}$$

$$\cancel{du = 8x-4\ dx}$$

Se puede factorizar el denominador?

No (a mano/calculadora)

$$4x^2-4x+3=(2x-1)^2+2$$

$x=\frac{u+1}{2}$

$u=2x-1$

$du=2dx$

$$I_2=\int \frac{x}{(2x-1)^2+2}\,dx$$

$$=\int \frac{\left(\frac{u+1}{2}\right)\frac{1}{2}dx}{u^2+2}=\frac{1}{4}\int \frac{u+1}{u^2+2}\,du$$

$$=\frac{1}{4}\int \frac{u}{u^2+2}\,du+\frac{1}{4}\int \frac{du}{u^2+2}$$

$$= \frac{1}{4}\frac{1}{2}\ln|u^2+2| + \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right) + C$$

$$I_3 := \int \frac{1\,dx}{4x^2-4x+3} = \int \frac{dx}{(2x-1)^2+2}$$

$u = 2x-1$

$du = 2dx$

$\frac{du}{2} = dx$

$$= \frac{1}{2}\int \frac{du}{u^2+2}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right)$$

Integral Original $= x + \frac{1}{4}\frac{1}{2}\ln|u^2+2| + \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right)$

$$- \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right)$$

$u = 2x-1$

$$= x + \frac{1}{8}\ln|(2x-1)^2+2| + \frac{1}{4\sqrt{2}}\arctan\left(\frac{2x-1}{\sqrt{2}}\right)$$

$$- \frac{1}{2\sqrt{2}}\arctan\left(\frac{2x-1}{\sqrt{2}}\right) + C$$