Clase 14: técnicas de integración

Recordemos: partes.

$$\int u\,dv = uv - \int v\,du$$

$$\int f\cdot g'\,dx = f\cdot g - \int g\cdot f'\,dx$$

Ej: $\int x e^{-x}\,dx = \int (-t)\,e^{t}\,(-dt)$

Sol: $t = -x$

$dt = -dx$

$= \int t e^{t}\,dt$

$$\int te^t\,dt = te^t - \int e^t\,dt$$

$u = t \qquad dv = e^t\,dt$

$du = dt \qquad v = e^t$

$$= te^t - e^t + C$$

Nos devolvimos a la variable x: $t = -x$

$$\int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

Otras técnicas de integración:

Sustitución trigonométrica.

Pregunta: $\int \cos^3(x)\,dx$??

Recordemos:

$$\operatorname{sen}^2(x) + \cos^2(x) = 1 \quad \text{muy útil}$$

$$\Rightarrow \quad \cos^2(x) = 1 - \operatorname{sen}^2(x) \quad / \cdot \cos(x)$$

$$\Rightarrow \quad \cos^3(x) = \cos(x)\left(1 - \operatorname{sen}^2(x)\right)$$

$$\int \cos^3(x)\,dx = \int \cos(x)\,dx - \int \cos(x)\operatorname{sen}^2(x)\,dx$$

$$= \operatorname{sen}(x) - \underbrace{\int \cos(x)\operatorname{sen}^2(x)\,dx}_{= I_1}$$

$$I_1 = \int \cos(x)\,\operatorname{sen}^2(x)\,dx \qquad \begin{cases} u = \operatorname{sen} x \\ du = \cos(x)\,dx \end{cases}$$

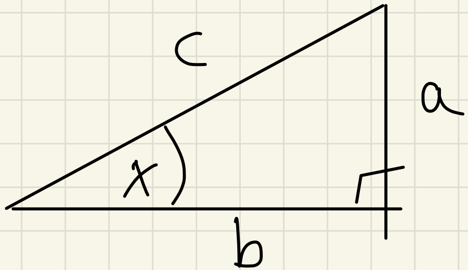
$$= \int u^2\,du = \frac{u^3}{3} + C$$

$$= \frac{(\operatorname{sen} x)^3}{3} + C$$

Volvemos a la integral original: $(\hat{C} = -C)$

$$\int \cos^3(x)\,dx = \operatorname{sen}(x) - \frac{(\operatorname{sen}(x))^3}{3} + \hat{C}$$

$$\operatorname{sen}^2(x) + \cos^2(x) = 1$$



$$a^2+b^2=c^2$$

$$\text{sen}(x)=\frac{a}{c}\ ,\ \cos(x)=\frac{b}{c}$$

$$\text{sen}^2(x)=\frac{a^2}{c^2}\ ;\ \cos^2(x)=\frac{b^2}{c^2}$$

$$\text{sen}^2(x)+\cos^2(x)=$$
$$\frac{a^2}{c^2}+\frac{b^2}{c^2}=\frac{a^2+b^2}{c^2}=\frac{c^2}{c^2}=1$$

Ej: $\int \text{sen}^5(x)\cos^2(x)\,dx$

$$\text{sen}^2(x)=1-\cos^2(x)$$
$$\text{sen}^4(x)=(1-\cos^2(x))^2$$

$$\text{sen}^5(x)=\text{sen}^4(x)\cdot\text{sen}(x)$$
$$=(1-\cos^2(x))^2\cdot\text{sen}(x)$$

Volviendo a la integral original

$$\int \operatorname{sen}^5(x)\cos(x)\,dx = \int (1-\cos^2(x))^2 \cdot \operatorname{sen}(x)\cdot \cos^2(x)\,dx$$

$$= \int (1-u^2)^2\, u^2 (-du) =$$

$$= -\int (1-u^2)^2 u^2\, du$$

$u = \cos(x)$

$du = -\operatorname{sen}(x)\,dx$

$$= -\int (1-2u^2+u^4)\, u^2\, du$$

$$= -\int u^2 - 2u^4 + u^6\, du$$

$u = \cos(x)$

$$= -\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C$$

$$= -\frac{\cos^3 x}{3} + \frac{2}{5}\cos^5(x) - \frac{\cos^7(x)}{7} + C$$

$\cos^2(x) = 1 - \operatorname{sen}^2(x)$

$\operatorname{sen}^2(x) = 1 - \cos^2(x)$

Ej: $\int_0^{\pi} \operatorname{sen}^2(x)\, dx$

$$= \int_0^{\pi} (1 - \cos^2(x))\,dx = \int_0^{\pi} 1\, dx - \int_0^{\pi} \cos^2(x)\, dx$$

$$= \int_0^{\pi} 1\, dx - \int_0^{\pi} (1 - \operatorname{sen}^2(x))\,dx$$

$$= \int_0^{\pi} \operatorname{sen}^2(x)\,dx$$

lo mismo con lo que empezamos!!

Hay que usar algo distinto: recordemos

$$\operatorname{sen}^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

$$\int_0^{\pi} \operatorname{sen}^2(x)\,dx = \int_0^{\pi} \frac{1}{2}(1 - \cos(2x))\,dx$$

$$= \int_0^{\pi} \frac{1}{2}\, dx - \underbrace{\int_0^{\pi} \cos(2x)\, dx}_{= I_1}$$

$$= \frac{\pi}{2}$$

$$I_1 = \int_0^{\pi} \cos(2x)\, dx$$

$$\int \cos(x)\, dx = \operatorname{sen}(x) + c$$

$$u = 2x \qquad x = 0 \Rightarrow u = 0$$

$$\frac{du}{2} = dx \qquad x = \pi \Rightarrow u = 2\pi$$

$$I_1 = \int_0^{2\pi} \cos(u)\, \frac{du}{2} = \frac{1}{2} \operatorname{sen}(u) \Big|_0^{2\pi} = 0$$

Nuestra integral original:

$$\int_0^{\pi} sen^2(x) = \frac{\pi}{2} - I_1 = \frac{\pi}{2}$$

Ej: $\int sen^4 x \, dx$

$$(sen^2(x))^2 = sen^4(x)$$

$$(sen^2(x))^2 = \left(\frac{1}{2}(1 - \cos(2x))\right)^2$$

$$= \frac{1}{4}\left(1 - 2\cos(2x) + \cos^2(2x)\right)$$

$$= \frac{1}{4}\left(1 - 2\cos(2x) + \frac{1}{2}(1 + \cos(4x))\right)$$

$$\int \operatorname{sen}^4(x)\,dx =$$

$$\int \frac{1}{4}\left(1 - 2\cos(2x) + \frac{1}{2}(1 + \cos(4x))\right)dx$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

$$\Downarrow \; x \mapsto 2x \qquad \Downarrow$$

$$\cos^2(2x) = \frac{1}{2}(1 + \cos(4x))$$

$$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \frac{1}{2} + \frac{1}{2}\cos(4x)\right)dx$$

$$= \frac{1}{4}\left(\int 1\,dx - \underbrace{\int 2\cos(2x)\,dx}_{I_1} + \int \frac{1}{2}\,dx + \underbrace{\int \frac{1}{2}\cos(4x)\,dx}_{I_2}\right)$$

$$= \frac{1}{4}\left(x - I_1 + \frac{x}{2} + I_2\right) + C$$

Calculamos I_1 e I_2

$$I_1 = \int 2\cos(2x)\,dx = \int \cos(u)\,du = \operatorname{sen}(u) + C$$

$u = 2x$

$du = 2\,dx$

$$= \operatorname{sen}(2x) + C$$

$$I_2 = \int \frac{1}{2}\cos(4x)\,dx = \int \frac{1}{2}\cos(u)\,\frac{du}{4}$$

$u = 4x$

$du = 4dx \Rightarrow$

$\frac{du}{4} = dx$

$$= \frac{1}{8}\int \cos(u)\,du$$

$$= \frac{1}{8}\,\text{sen}(u) + C$$

$$= \frac{1}{8}\,\text{sen}(4x)$$

Finalmente, nuestra integral es

$$\int \text{sen}^4 x\,dx = \frac{1}{4}\left(x - \text{sen}(2x) + \frac{x}{2} + \frac{1}{8}\,\text{sen}(4x)\right) + C$$

Ej: $\int \tan^6 x \sec^4 x \, dx$

$$= \int \tan^6 x \cdot \sec^2 x \cdot \sec^2 x \, dx$$

$$= \int \tan^6 x \cdot (\tan^2 x + 1) \cdot \sec^2 x \, dx$$

$$= \int u^6 (u^2 + 1) \, du$$

$$= \int u^8 + u^6 \, du$$

$$= \frac{u^9}{9} + \frac{u^7}{7} + C$$

$$\operatorname{sen}^2 x + \cos^2 x = 1 \quad / \frac{1}{\cos^2 x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\frac{\operatorname{sen}^2 x}{\cos^2 x} + 1 = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$u = \tan x$$

$$du = \sec^2 x \, dx$$

$$= \frac{\tan^9 x}{9} + \frac{\tan^7 x}{7} + C$$

Ej: $\int \tan^3 x \, dx$

$\tan^2 x + 1 = \sec^2 x$

$\frac{d}{dx} \tan x = \sec^2 x$

$$= \int \tan x \cdot \tan^2 x \, dx$$

$$= \int \tan x (\sec^2 x - 1) \, dx$$

$$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

$u = \tan x \quad du = \sec^2 x \, dx$

$$= \int u\,du - \int \tan x\,dx$$

$$= \frac{u^2}{2} - \int \tan x\,dx + C$$

$$= \frac{\tan^2 x}{2} - \int \tan x\,dx + C$$

$$= \frac{\tan^2 x}{2} + \ln|\cos x| + C$$

$$\int \frac{\operatorname{sen} x}{\cos x}\,dx$$

$u = \cos x$
$du = -\operatorname{sen} x\,dx$

$$= \int \frac{-du}{u} = -\ln|u|$$

$$= -\ln|\cos x|$$

Ej: $\int \operatorname{sen}(4x)\cos(5x)\,dx$

Sol: $\operatorname{sen} A \cdot \cos B = \frac{1}{2}[\operatorname{sen}(A-B) + \operatorname{sen}(A+B)]$

$A = 4x$
$B = 5x$

$$\operatorname{sen}(4x) \cdot \cos(5x) = \frac{1}{2}(\operatorname{sen}(-x) + \operatorname{sen}(9x))$$

$$\operatorname{sen}(-x) = -\operatorname{sen}(x)$$

$$\int \operatorname{sen}(4x)\cos(5x)dx = \frac{1}{2}\int -\operatorname{sen}(x) + \operatorname{sen}(9x)\,dx$$

$$= \frac{1}{2}\int -\operatorname{sen} x\,dx + \frac{1}{2}\int \operatorname{sen}(9x)\,dx$$

$$= \frac{1}{2} \cdot (\cos(x)) + \frac{1}{2} \cdot \frac{1}{9} - \cos(9x) + C$$

$$\int \operatorname{sen}(9x)\,dx \overset{\substack{u = 9x \\ du = 9\,dx}}{=} \int \operatorname{sen}(u)\frac{du}{9} = -\frac{\cos(u)}{9}$$

$$= -\frac{1}{9}\cos(9x)$$

Ej: $\int \sec^3 x \, dx$

Pista: por partes.

Pista 2: $\sec^3 x = \sec x \cdot \sec^2 x$